Q-PSEUDODIFFERENCE DRINFELD-SOKOLOV REDUCTION FOR ALGEBRA OF COMPLEX SIZE MATRICES.

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ABSTRACT. The q-deformed version of the Drinfeld-Sokolov reduction is extended to the case of the algebra of 'complex size matrices'; this construction generalizes earlier results of B.Khesin and F.Malikov on universal DS reduction and follows the pattern of recent studies of q-deformed DS reduction which were started by E.Frenkel, N.Reshetikhin and one of the authors.

1. INTRODUCTION

It is well-known that the space $\mathcal{M}_n$ of scalar n-th order differential operators has a remarkable quadratic Poisson structure, called the (second) Adler-Gelfand-Dickey bracket [1, 9]. This structure admits several different realizations. The first one, known as the Drinfeld-Sokolov reduction [2], shows that the Gelfand-Dickey bracket can be obtained via Hamiltonian reduction from a linear Poisson bracket on the space of matrix first order differential operators which is considered as the dual space of the affine algebra $\widehat{\mathfrak{sl}}_n$. This construction has a natural generalization to the case of an arbitrary semisimple Lie algebra $\mathfrak{g}$, the corresponding Poisson algebra $\mathcal{W}(\mathfrak{g})$ of functionals on reduced space is called the classical $\mathcal{W}$-algebra (associated with $\mathfrak{g}$).

The second realization is based on the study of the Lie group of integral operators (more precisely, of some its extension, see [10]). This group comes equipped with the natural Sklyanin bracket which endows it with the structure of a Poisson-Lie group, and the second Gelfand-Dickey bracket is identified with the restriction of this bracket to the subvariety $\mathcal{M}_n$, which is Poisson in this case.

This construction leads to a natural generalization of the Gelfand-Dickey bracket for the space of pseudodifferential symbols of any complex degree λ

$$G_\lambda = \left\{ \partial^\lambda + u_1(x)\, \partial^{\lambda-1} + \dots \right\}.$$

At the same time, DS-reduction [2] is defined only for $\lambda = n$, $n \in \mathbb{N}$, and only for the Poisson subspace $\mathcal{M}_n \subset G_n$. B.Khesin and F.Malikov in [11] proposed a counterpart of the DS-reduction which applies to all pseudodifferential symbols of complex degree. To describe their construction (called the universal DS-reduction) let us recall the definition of the algebras $\mathfrak{gl}_\lambda$ of complex size matrices.

The definition of "complex size matrices" was proposed by B.Feigin, who used them to compute the cohomology of the Lie algebra of differential operators, see [3].

Consider the universal enveloping algebra $U(\mathfrak{sl}_2)$; it has the well-known Casimir element $C = ef + fe + \frac{1}{2}h^2$, where e, f, h — standard basis of $\mathfrak{sl}_2$. By definition,

$$\mathfrak{gl}_\lambda = U(\mathfrak{sl}_2) / \left\{ C = \frac{1}{2}(\lambda - 1)(\lambda + 1) \right\}.$$

For positive integer $\lambda = n$ the algebra $\mathfrak{gl}_\lambda$ contains a large ideal, and the quotient algebra is isomorphic to $\mathfrak{gl}_n$, which explains the name "algebra of complex size matrices".

More precisely, to perform the universal DS-reduction we need not the algebra $\mathfrak{gl}_\lambda$ itself, but some its completion $\bar{\mathfrak{gl}}_\lambda$. B.Khesin and F.Malikov [11] proved that the Gelfand-Dickey bracket on G_λ can be obtained by reduction from a linear bracket on the dual of the affine Lie algebras corresponding to $\bar{\mathfrak{gl}}_\lambda$. (This was conjectured earlier by B.Feigin and C.Roger).

The third realization of Gelfand-Dickey brackets is based on the study of the center of the quantized enveloping algebra $U_q\left(\widehat{\mathfrak{sl}}_n\right)$ at the critical value of the central charge, see [4]. We shall not consider it here, although it is this realization which was generalized for the first time to q-difference setting and allowed to construct the q-deformed $\mathcal{W}$-algebras (see [7]).

A q-difference version of the DS reduction was defined in [8] for the $\mathfrak{sl}_2$ case and generalized to the case of arbitrary semisimple algebra in [17]. The consistency conditions for this reduction lead to a new class of elliptic r-matrices which are fixed by these conditions in an essentially unique way (one for each semisimple Lie algebra).

The q-difference counterpart of the second construction of the Gelfand-Dickey brackets was proposed in [14]. It is based on the study of the group of q-pseudodifference symbols of arbitrary complex degrees

$$\widehat{\mathbb{G}}_- = \bigcup_{\lambda \in \mathbb{C}} \widehat{\mathbb{G}}_\lambda, \quad \widehat{\mathbb{G}}_\lambda = \left\{ L = D^\lambda + u_1(z) D^{\lambda-1} + \cdots \right\},$$

where D is the dilation operator, $(Df)(z) = f(qz)$. It was shown that in a natural class of r-matrix Poisson brackets on $\widehat{\mathbb{G}}_\lambda$ there exists a unique one with respect to which formal spectral invariants $H_n(L) = \frac{\lambda}{n} \mathrm{Tr}\, L^{\frac{n}{\lambda}}$ are in involution. (These spectral invariants give rise to the generalized q-deformed KdV hierarchy $\frac{dL}{dt} = \left[L_{(+)}^{\frac{m}{\lambda}}, L \right]$ described earlier in [6].) This bracket was called the generalized q-deformed Gelfand-Dickey structure. Similarly to the differential case, for positive integer $\lambda = n$ the subspace of q-difference operators of n-th order

$$\mathbb{M}_n = \left\{ L = D^n + u_1(z) D^{n-1} + \cdots + u_0(z) \right\}$$

is Poisson. The Poisson algebra of the functionals on $\mathbb{M}_n$ coincides (up to the constraint $u_0(z) = 1$) with the q-deformed $\mathcal{W}$-algebra $\mathcal{W}_q(\mathfrak{sl}_n)$, constructed in [7] and [8, 17].

In the present paper we generalize to the q-difference setting the procedure of the universal DS-reduction [11]. We define an algebra $\mathfrak{gl}_q$ consisting of $\mathfrak{gl}_\infty$-matrices of special form whose matrix elements are holomorphic functions of a complex variable t. (This algebra is an extension of the algebra $\mathfrak{gl}$, which was also constructed in [11]). There exists a natural evaluation map $i_\lambda : \mathfrak{gl}_q \mapsto \mathfrak{gl}_\infty$, which assigns to a matrix-function $A \in \mathfrak{gl}_q$ its value at an arbitrary fixed point $t = \lambda$: $A \mapsto A(\lambda)$. The image of this evaluation map is a subalgebra in $\mathfrak{gl}_\infty$ which will be called (extended) algebra of complex size matrices (more precisely, of size $\lambda \times \lambda$. In fact, we have a family of such algebras parametrized by a complex number λ). Due to the infinite dimension of $\mathfrak{gl}_q^\lambda$, the definition of the corresponding loop algebras $L\mathfrak{gl}_q^\lambda$ involves some peculiarities.

Like in the $\mathfrak{sl}_n$-case [8, 17], the reduction procedure consists of two steps: first, we impose constraints fixing a submanifold $\mathbb{Y}_q^\lambda \subset L\mathfrak{gl}_q^\lambda$ which is preserved by the q-deformed gauge action of the upper-triangular group $L\mathfrak{N}_+$, and then we take the quotient over this group. We prove that the quotient $\mathbb{Y}_q^\lambda / L\mathfrak{N}_+$ can be identified with the space $\widehat{\mathbb{G}}_\lambda$ of q-pseudodifference symbols of degree λ.

Using a method similar to the one of [8, 17], we describe explicitly all r-matrix Poisson brackets on $L\mathfrak{gl}_q^\lambda$ (in a wide natural class) which admit the q-deformed universal DS-reduction. At this point we encounter a new phenomenon. Recall, that in the $\mathfrak{sl}_n$-case the underlying classical r-matrix was related with the decomposition of the algebra $L\mathfrak{sl}_n$ into the sum of the subalgebras of upper-triangular, lower-triangular and diagonal matrices. Its diagonal part $\hat{r}^0$ was given by the Cayley transformation of the operator $D\tau_n$, the operator τ_n acting by *cyclic permutation* of matrix elements. Obviously, in the $\mathfrak{gl}_q^\lambda$-case there exists no analogue of τ_n. We shall see that it must be replaced by *the shift operator $\hat{s}$* whose properties are quite different. This causes some difficulties in the definition of the diagonal part of the r-matrix, which requires a regularization introducing some free parameters into the admissible r-matrix; the uniqueness is restored, however, if we demand the formal spectral invariants to be in involution, and the resulting quotient Poisson structure coincides precisely with the generalized q-deformed Gelfand-Dickey structure defined in [14].

This article has the following structure. In section 2 we recall in more details the finite-dimensional DS-reduction [8, 17], as well as some results of [14] and generalize them to the $\mathfrak{gl}_n$-case. Unlike the $\mathfrak{sl}_n$-case, we obtain a family of Poisson structures admitting reduction; however, only one of them gives rise to a quotient bracket satisfying the involutivity condition. This result is a finite-dimensional analogue of the uniqueness theorem for $\mathfrak{gl}_q^\lambda$ and will be used to prove the latter.

In section 3 we construct algebras $\mathfrak{gl}_q$, $\mathfrak{gl}_q^\lambda$ and their loop algebras. Section 4 is devoted to the cross-section theorem which gives a model of the quotient

$\mathbb{Y}_q^\lambda / L\mathfrak{N}_+$. In section 5 we describe explicitly all r-matrix Poisson brackets on $L\mathfrak{gl}_q^\lambda$, which admit the q-deformed universal DS-reduction. Section 6 is devoted to the uniqueness theorem.

Throughout the article we shall use the following notation. We fix a complex number q, $|q| < 1$. Let $\hat{h}$ be the dilation operator,

$$\hat{h}a(z) = a(qz), \quad a \in \mathbb{C}\left(\left(z^{-1}\right)\right), \quad q \in \mathbb{C}, \quad |q| < 1. \tag{1.1}$$

We shall denote the same dilation operator by D when it is considered as a generator of the algebra of q-pseudodifference operators (see below).

We fix the branch of $\ln w$, $w \in \mathbb{C}$, by

$$-\pi < \arg w < \pi, \quad \ln 1 = 0,$$

and put $q^w \equiv \exp(w \ln q)$. Arbitrary complex degrees of $\hat{h}$ are defined by

$$^{h^w}a(z) \equiv \left(\hat{h}^w a\right)(z) = a\left(q^w z\right), \quad \forall w \in \mathbb{C}, \quad a \in \mathbb{C}\left(\left(z^{-1}\right)\right). \tag{1.2}$$

For $a \in \mathbb{C}\left(\left(z^{-1}\right)\right)$, $a = \sum_i a_i z^i$, we put

$$\int a(z)dz/z = Res\, a = a_0; \tag{1.3}$$

clearly, this formal integral is dilation invariant, i.e.,

$$\int a(z)dz/z = \int a(qz)dz/z. \tag{1.4}$$

We introduce an $\hat{h}$-invariant inner product in $\mathbb{C}\left(\left(z^{-1}\right)\right)$ by

$$\langle a, b \rangle_\mathbb{C} = \int \frac{dz}{z} a(z)b(z). \tag{1.5}$$

Let $\mathfrak{a}$ be a linear space with an inner product $\langle \cdot, \cdot \rangle$. We shall denote by $\langle\langle \cdot, \cdot \rangle\rangle$ the following inner product on $\mathfrak{a} \oplus \mathfrak{a}$:

$$\left\langle\left\langle \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \right\rangle\right\rangle = \langle X_1, Y_1 \rangle - \langle X_2, Y_2 \rangle. \tag{1.6}$$

2. The q-deformed DS-reduction in $\mathfrak{gl}_n$: overview of the results [8, 17, 14] and some generalizations.

2.1. **Reduction procedure and the choice of r-matrix.** In this subsection we recall briefly the procedure of DS-reduction and describe all the r-matrix Poisson brackets on $\mathfrak{gl}_n$ (in a wide natural class) which admit this reduction. Unlike the $\mathfrak{sl}_n$-case where this bracket is essentially unique (see [8, 17]), there exists a family of such brackets parametrized by a skew-symmetric operator in $\mathbb{C}\left(\left(z^{-1}\right)\right)$. This non-uniqueness does not lead to a new kind of deformed Gelfand-Dickey brackets, since, as we shall see in subsection 2.2, only one of the quotient Poisson structures satisfies the involutivity condition.

We fix the following notation. Let $\mathfrak{n}_+(n)$, $\mathfrak{n}_-(n)$, $\mathfrak{h}_n \subset \mathfrak{gl}_n$ be the subalgebras of strictly upper triangular, strictly lower triangular and diagonal matrices, respectively, let $\mathfrak{b}_\pm(n) = \mathfrak{n}_\pm(n) \oplus \mathfrak{h}_n$ be the subalgebras of upper (lower) triangular matrices with arbitrary diagonal elements; let $\mathbf{N}_+(n) \subset \mathfrak{gl}_n$ be the unipotent group corresponding to $\mathfrak{n}_+(n)$. We shall denote by $L\mathfrak{gl}_n$, $L\mathfrak{n}_\pm(n)$, $L\mathfrak{b}_\pm(n)$, $L\mathfrak{h}_n$, $L\mathbf{N}_+(n)$ the corresponding loop algebras (group).

We introduce an invariant inner product on $L\mathfrak{gl}_n$ by

$$\langle A, B \rangle = \int \frac{dz}{z} \mathrm{Tr} A(z) B(z), \quad A, B \in L\mathfrak{gl}_n. \tag{2.1}$$

We shall denote by $\mathbb{M}_n$ the space of scalar n-th order q-difference operators of the form

$$L = D^n + u_1(z)D^{n-1} + \cdots + u_n(z), \quad u_i \in \mathbb{C}\left(\left(z^{-1}\right)\right). \tag{2.2}$$

Now let us briefly recall the DS-reduction procedure. It is well known that a scalar q-difference equation of order n

$$L\psi_0 = 0, \quad L \in \mathbb{M}_n,$$

is equivalent to a first order matrix equation

$$D\Psi = \mathbb{L}\Psi, \quad \Psi = \begin{pmatrix} \psi_{n-1} \\ \vdots \\ \psi_0 \end{pmatrix},$$

where the potential $\mathbb{L} \in L\mathfrak{gl}_n$ has a special form. The standard choice for $\mathbb{L}$ is given by a companion matrix:

$$\mathbb{L} = \begin{pmatrix} -u_1 & \cdots & -u_{n-1} & -u_n \\ 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{2.3}$$

This choice is not unique; a linear change of variables

$$\Psi \mapsto \Psi' = S\Psi, \quad S \in L\mathbf{N}_+(n)$$

induces a gauge transformation

$$\mathbb{L} \longmapsto \mathbb{L}' = {}^h S \mathbb{L} S^{-1}. \tag{2.4}$$

Let us denote by $\mathbb{Y}_n \subset L\mathfrak{gl}_n$ the subvariety of all matrices of the form $\mathbb{L}' = \Lambda_n + A$, $A \in L\mathfrak{b}_+(n)$, where

$$\Lambda_n = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{2.5}$$

It is easy to see that the gauge action (2.4) of the group $LN_+(n)$ preserves $\mathbb{Y}_n$.

Theorem 2.1. [8, 17]

1. *The gauge action of $LN_+(n)$ on $\mathbb{Y}_n$ is free.*
2. *The set of companion matrices of the form (2.3) is a cross-section of this action, so the quotient $\mathbb{Y}_n/LN_+(n)$ can be identified with $\mathbb{M}_n$.*

The quotient $\mathbb{Y}_n/LN_+(n)$ has a natural description in the framework of Poisson reduction proposed in [8, 17].

First of all note that the set $L\mathfrak{gl}_n$ of all matrix first order difference operators with potential of "general form"

$$\mathbb{A} = D - \mathbb{L}, \quad \mathbb{L} \in L\mathfrak{gl}_n,$$

can be supplied with the structure of a Poisson manifold. Unlike the differential case, the choice of this structure is not quite unique . However, it may be fixed in a canonical way if we supply the gauge group with the structure of a Poisson Lie group and require the Poisson bracket on $\mathcal{M}_n$ to be *covariant* with respect to the gauge action; in other words, the map

$$L\,GL_n \times L\mathfrak{gl}_n \to L\mathfrak{gl}_n: \quad \mathbb{L} \longmapsto \mathbb{L}' = {}^hS\mathbb{L}S^{-1}.$$

has to be a Poisson mapping [15]. The Poisson bracket on $L\mathfrak{gl}_n$ is explicitly described in terms of the r-matrix $\hat{r}$ which fixes the Poisson structure on gauge group (this r-matrix must satisfy the natural invariance condition $\left(\hat{h} \otimes \hat{h}\right)\hat{r} = \hat{r}$). We shall write down the corresponding formula a little later after we introduce some necessary notations (see (2.6 below). The reduction is actually performed with respect to a subgroup of the full gauge group; the natural consistency conditions are as follows:

First, the invariants of the gauge action of $LN_+(n)$ on $L\mathfrak{gl}_n$ must form a Lie subalgebra $\mathcal{I}$ with respect to the Poisson bracket on $L\mathfrak{gl}_n$; (in that case the subgroup $LN_+(n)$ is said to be an admissible subgroup of the full gauge group (regarded as a Poisson Lie group)). Second, the constraints defining the submanifold $\mathbb{Y}_n \subset L\mathfrak{gl}_n$ must generate a Poisson ideal in $\mathcal{I}$.

The latter condition means that the Poisson brackets of the constraints vanish on the constraints surface $\mathbb{Y}_n$, i.e., the constraints are of the *first class,* according to Dirac. Both conditions impose restrictions on the choice of the initial r-matrix (which eventually allow to fix it completely).

To describe the relevant Poisson brackets explicitly let us fix the following notation.

By definition, a functional $\widehat{\varphi} \in Fun\,(L\mathfrak{gl}_n)$ is said to be smooth if for any $\mathbb{L} \in L\mathfrak{gl}_n$ there exists an element $d\widehat{\varphi}(\mathbb{L}) \in L\mathfrak{gl}_n$ called its linear gradient such that

$$\langle d\widehat{\varphi}(\mathbb{L}), X \rangle = \left(\frac{d}{dt}\right)_{t=0} \varphi(\mathbb{L} + tX), \quad \forall X \in L\mathfrak{gl}_n.$$

In applications, various functionals may be defined only on an affine subspace of $L\mathfrak{gl}_n$; in that case the choice of the gradient (when it exists) is not unique (however, a canonical choice is frequently possible).

For a smooth functional $\widehat{\varphi}$ we define its left and right gradients by $\nabla\widehat{\varphi}(\mathbb{L}) = \mathbb{L}d\widehat{\varphi}(\mathbb{L})$ and $\nabla'\widehat{\varphi}(\mathbb{L}) = d\widehat{\varphi}(\mathbb{L})\mathbb{L}$, respectively.

Let $\hat{r} \in End(L\mathfrak{gl}_n)$ be a classical r-matrix. We assume that $\hat{r}$ is skew symmetric and satisfies the modified classical Yang-Baxter equation and, moreover, $\hat{r} \circ \hat{h} = \hat{h} \circ \hat{r}$. Put $\hat{r}_{\pm} = \hat{r} \pm \frac{1}{2}id$. The natural Poisson bracket on $L\mathfrak{gl}_n$ which is covariant with respect to gauge transformations is given by

$$\left\{\hat{\varphi}, \hat{\psi}\right\} = \left\langle\!\left\langle\!\left\langle \begin{pmatrix} \hat{r} & -\hat{h}\hat{r}_{+} \\ \hat{r}_{-}\hat{h}^{-1} & -\hat{r} \end{pmatrix} \begin{pmatrix} \nabla\hat{\varphi} \\ \nabla'\hat{\varphi} \end{pmatrix}, \begin{pmatrix} \nabla\hat{\psi} \\ \nabla'\hat{\psi} \end{pmatrix}\right\rangle\!\right\rangle\!\right\rangle \qquad (2.6)$$

(see [15], where this formula is derived from the theory of the so called *twisted Heisenberg double*). In the present context we need to choose $\hat{r}$, so as to assure the admissibility of $L\mathbf{N}_{+}(n)$. The admissibility criterion may be found in [15]; here we only describe the r-matrices which make $L\mathbf{N}_{+}(n)$ admissible.

Let $\mathcal{P}_{+}, \mathcal{P}_{-}, \mathcal{P}_0$ be the projection operators onto $L\mathfrak{n}_{+}(n), L\mathfrak{n}_{-}(n), L\mathfrak{h}_n$, respectively. We denote by $\mathcal{P}_{00}$ the orthogonal projection operator onto the one-dimensional subspace $\mathbb{C} \cdot 1 \subset L\mathfrak{h}_n$ and put $\mathcal{P}'_0 = \mathcal{P}_0 - \mathcal{P}_{00}$.

Put

$$\hat{r} = \frac{1}{2}(\mathcal{P}_{+} - \mathcal{P}_{-}) + \hat{r}^0\mathcal{P}_0, \qquad (2.7)$$

where $\hat{r}^0 \in End(L\mathfrak{h}_n)$, $\hat{r}^0 = -(\hat{r}^0)^*$.

It may be shown that with this choice of $\hat{r}$ the gauge action of $L\mathbf{N}_{+}(n)$ is admissible.

The remaining freedom in choice of $\hat{r}$ may be (almost) eliminated when we impose our second condition. Namely, let $\hat{\tau}_n \in End(L\mathfrak{h}_n)$ be the operator acting by the cyclic permutation of matrix elements:

$$\hat{\tau}_n\mathrm{diag}(f_0, f_1, \ldots, f_{n-1}) = \mathrm{diag}(f_1, \ldots, f_{n-1}, f_0). \qquad (2.8)$$

Let $U_n \subset L\mathfrak{h}_n$ be the subspace of matrices of the form

$$U_n = \left\{\mathrm{diag}\left(f_0(z), f_0(q^{-1}z), \ldots, f_0(q^{-(n-1)}z)\right), \quad f_0 \in \mathbb{C}\left(\!\left(z^{-1}\right)\!\right)\right\}. \qquad (2.9)$$

Theorem 2.2. *The Poisson bracket of the form (2.6) admits the q-deformed DS-reduction if and only if the corresponding operator $\hat{r}^0$ has the form:*

$$\hat{r}^0 = \frac{1}{2}\frac{1 + \hat{h}\hat{\tau}_n}{1 - \hat{h}\hat{\tau}_n}\mathcal{P}'_0 + \Delta\mathcal{P}_{U_n} + \hat{\alpha} - \hat{\alpha}^*, \qquad (2.10)$$

where $\mathcal{P}_{U_n}$ is the orthogonal projection operator onto U_n, Δ is a skew symmetric operator in U_n and $\hat{\alpha}$ is a linear operator of the form

$$\hat{\alpha}\left(f\right) = \langle f, \alpha \rangle \cdot 1 \subset L\mathfrak{h}_n, \quad \alpha \in U_n^\perp. \qquad (2.11)$$

Remark 2.1. *The operator* $1 - \hat{h}\hat{\tau}_n$ *is not invertible in* $L\mathfrak{h}_n$. *It is easy to see that* $\mathrm{Ker}\left(1 - \hat{h}\hat{\tau}_n\right) = \mathbb{C} \cdot 1$, $\mathrm{Im}\left(1 - \hat{h}\hat{\tau}_n\right) = L\mathfrak{h}_n'$, *where*

$$L\mathfrak{h}_n' \equiv L\mathfrak{h}_n \ominus \{\mathbb{C} \cdot 1\} = \mathcal{P}_0'\left(L\mathfrak{h}_n\right). \qquad (2.12)$$

Hence we may define a regularized inverse operator $\left(1 - \hat{h}\hat{\tau}_n\right)^{-1} : L\mathfrak{h}_n' \to L\mathfrak{h}_n'$. *In* (2.10) $\mathcal{P}_0'$ *is the projection operator onto* $L\mathfrak{h}_n'$, *so this expression is well-defined.*

Remark 2.2. *The bracket constructed in* [8, 17] *corresponds to*

$$\hat{r}_{0,n}^0 \equiv \frac{1}{2} \frac{1 + \hat{h}\hat{\tau}_n}{1 - \hat{h}\hat{\tau}_n} \mathcal{P}_0'. \qquad (2.13)$$

Proof. Denote by $V_n \subset L\mathfrak{h}_n$ the space of matrices of the form:

$$V_n = \{\mathrm{diag}\left(0, *, \dots, *\right)\}.$$

Lemma 2.3. *Condition 2) above is equivalent to the following equation for* $\hat{r}^0$:

$$\hat{r}^0 \left(1 - \hat{h}\hat{\tau}_n\right) f = \frac{1}{2}\left(1 + \hat{h}\hat{\tau}_n\right) f + \tilde{\alpha}\left(f\right), \quad \forall f \in V_n, \qquad (2.14)$$

where $\tilde{\alpha}$ *is a linear operator in* $L\mathfrak{h}_n$ *with* $\mathrm{Im}\tilde{\alpha} \subset \mathbb{C} \cdot 1 \subset L\mathfrak{h}_n$.

We omit the proof, since it is similar to the proof of proposition 5.1 below.
Let us define the following subspaces in $L\mathfrak{h}_n$:

$$\begin{aligned} U_n' &= L\mathfrak{h}_n' \cap U_n; \\ \mathrm{Im}V_n &= \left(1 - \hat{h}\hat{\tau}_n\right) V_n. \end{aligned} \qquad (2.15)$$

It is easy to see that

$$L\mathfrak{h}_n = U_n \oplus \mathrm{Im}V_n = \mathbb{C} \cdot 1 \oplus U_n' \oplus \mathrm{Im}V_n. \qquad (2.16)$$

The following lemma finishes our arguments:

Lemma 2.4. *Any skew-symmetric operator* $\hat{r}^0 \in EndL\mathfrak{h}_n$ *satisfying* (2.14) *has the form* (2.10).

⊙ Put $\mathcal{D} = \hat{r}^0 - \hat{r}_{0,n}^0$. It is evident that $\hat{r}_{0,n}^0$ satisfies (2.14) with $\tilde{\alpha} = 0$ (because $\mathbb{C} \cdot 1 \perp \mathrm{Im}V_n$), hence $\mathcal{D}$ satisfies

$$\tilde{\alpha}\left(f\right) = \mathcal{D}\left(1 - \hat{h}\hat{\tau}_n\right) f, \quad \forall f \in V_n. \qquad (2.17)$$

We put $\hat{\alpha} = \tilde{\alpha} \left(1 - \hat{h}\hat{\tau}_n \right)^{-1}$ and rewrite (2.17) as

$$(\mathcal{D} - \hat{\alpha}) \, \bar{f} = 0, \quad \forall \bar{f} \in \mathrm{Im}V_n. \tag{2.18}$$

The operator $\hat{\alpha}$ is defined only on the subspace $\mathrm{Im}V_n \approx (\mathrm{Im}V_n)^*$, hence it can be written in the form $\hat{\alpha} = \langle \cdot, \alpha \rangle \cdot 1$ with some $\alpha \in \mathrm{Im}V_n = U_n^\perp$.

Skew-symmetry of $\mathcal{D}$ implies that it has the following block form with respect to the orthogonal decomposition $L\mathfrak{h}_n = \mathbb{C} \cdot 1 \oplus U_n' \oplus \mathrm{Im}V_n$:

	$\mathbb{C} \cdot 1$	U_n'	$\mathrm{Im}V_n$
$\mathbb{C} \cdot 1$	0	β	γ
U_n'	$-\beta^*$	a	b
$\mathrm{Im}V_n$	$-\gamma^*$	$-b^*$	d

$$a = -a^*,$$
$$d = -d^*. \tag{2.19}$$

The equation (2.18) implies $\gamma = \hat{\alpha}$, $b = d = 0$. Put

$$\Delta = \begin{pmatrix} 0 & \beta \\ -\beta^* & a \end{pmatrix} \in End\,(U_n),$$

then $\mathcal{D} = \Delta \mathcal{P}_{U_m} + \hat{\alpha} - \hat{\alpha}^*$, as desired. ∎

Remark 2.3. *It is easy to see that the term $\hat{\alpha} - \hat{\alpha}^*$ does not affect the value of the reduced bracket.*

Indeed, let us denote

$$Z_{\hat{\varphi}} = {}^{h^{-1}}\nabla\hat{\varphi} - \nabla'\hat{\varphi}, \quad \bar{Z}_{\hat{\varphi}} = {}^{h^{-1}}\nabla\hat{\varphi} + \nabla'\hat{\varphi}. \tag{2.20}$$

The bracket (2.6) may be written as

$$\left\{ \widehat{\varphi}, \widehat{\psi} \right\} = \left\langle Z_{\hat{\varphi}}, \tfrac{1}{2}\bar{Z}_{\hat{\psi}} - \hat{r}Z_{\hat{\psi}} \right\rangle. \tag{2.21}$$

The formula (2.21) implies that the contribution p_α of the term $\hat{\alpha} - \hat{\alpha}^*$ into the bracket is given by

$$p_\alpha = \left\langle \hat{\alpha}\left(Z_{\hat{\varphi}} \right), Z_{\hat{\psi}} \right\rangle - \left\langle \hat{\alpha}\left(Z_{\hat{\psi}} \right), Z_{\hat{\varphi}} \right\rangle = \hat{\alpha}\left(Z_{\hat{\varphi}} \right) Tr \, Z_{\hat{\psi}} - \hat{\alpha}\left(Z_{\hat{\psi}} \right) Tr \, Z_{\hat{\varphi}} = 0,$$

since $Tr \, Z_{\hat{\psi}} = Tr \, Z_{\hat{\varphi}} = 0$, due to invariance of the inner product.
Below we always put $\hat{\alpha} = 0$.

Now we calculate the U_n-block of the r-matrix $\hat{r}_{0,n}^0$. This auxiliary result will be used in section 6 to compute the quotient bracket obtained via DS-reduction from the algebra of complex size matrices.

Theorem 2.5.

$$\left\langle \hat{r}_{0,n}^0 \bar{f}, \bar{g} \right\rangle = \left\langle \frac{n}{2}\frac{1 + \hat{h}^n}{1 - \hat{h}^n}\mathcal{P}_0'\bar{f}, \bar{g} \right\rangle, \quad \forall \bar{f}, \bar{g} \in U_n \tag{2.22}$$

Proof. To calculate the l.h.s. we expand $\bar{f}, \bar{g}$ with respect to the basis of eigenfunctions of the operator $\hat{h}\hat{\tau}_n$.

Lemma 2.6. *The eigenfunctions of the operator $\hat{h}\hat{\tau}_n$ are*

$$E_{m,\alpha} = z^m \mathbf{e}_\alpha, \quad m \in \mathbb{Z}, \quad \alpha = 0, \dots, n-1, \tag{2.23}$$

where

$$\mathbf{e}_\alpha = \mathbf{diag}\left(1, \omega^\alpha, \dots, \omega^{(n-1)\alpha}\right), \quad \omega = e^{\frac{2\pi i}{n}}. \tag{2.24}$$

The corresponding eigenvalues $\xi_{m,\alpha}$ are equal to

$$\xi_{m,\alpha} = q^m \omega^\alpha. \tag{2.25}$$

The eigenfunctions satisfy

$$\langle E_{m,\alpha}, E_{l,\beta} \rangle = n\delta_{m,-l} \cdot \begin{cases} 1, & \alpha = -\beta \,mod\, n, \\ 0, & in\ other\ cases, \end{cases} \tag{2.26}$$

and form a basis in $L\mathfrak{h}_n$.

We shall denote by $\sum'_{m,\alpha}$ the sum over all pairs $(m, \alpha) \neq (0,0)$, $m \in \mathbb{Z}$, $\alpha = 0, \dots, n-1$. Note that in the expansion of $\bar{f}$ with respect to the eigenbasis $E_{m,\alpha}$ the $E_{0,0}$-component is annihilated by $\mathcal{P}'_0$, hence

$$\langle \hat{r}^0_{0,n} \bar{f}, \bar{g} \rangle = \sum'_{m,\alpha} \frac{1}{n} \frac{1 + q^m \omega^\alpha}{1 - q^m \omega^\alpha} \langle \bar{f}, E_{-m,n-\alpha} \rangle \langle \bar{g}, E_{m,\alpha} \rangle. \tag{2.27}$$

Any element of U_n has the form $\bar{f} = \mathrm{diag}\left(f(z), f(q^{-1}z), \dots, f(q^{-(n-1)}z)\right)$; we denote by f_m the coefficient of the formal Laurent expansion of $f(z)$ corresponding to z^m: $f(z) = \sum_{m=-\infty}^{N(f)} f_m z^m$. It is easy to see that

$$\langle \bar{f}, E_{-m,n-\alpha} \rangle = f_m \sum_{i=0}^{n-1} (q^m \omega^\alpha)^{-i} = f_m \frac{1 - q^{-mn}\omega^{-\alpha n}}{1 - q^{-m}\omega^{-\alpha}} = f_m \frac{1 - q^{-mn}}{1 - q^{-m}\omega^{-\alpha}}. \tag{2.28}$$

Substituting (2.28) and a similar expression for $\langle \bar{g}, E_{m,\alpha} \rangle$ into (2.27) we find:

$$\langle \hat{r}^0_{0,n} \bar{f}, \bar{g} \rangle = \frac{1}{2} \sum'_{m,\alpha} f_m g_{-m} (1 - q^{mn})^2 q^{-mn} \frac{1}{n} \frac{1 + q^m \omega^\alpha}{(1 - q^m \omega^\alpha)^3} q^m \omega^\alpha. \tag{2.29}$$

In the sum above the terms corresponding to $m = 0$ vanish due to the multiplier $1 - q^{mn}$.

Lemma 2.7.

$$S_0 \equiv \sum_{\alpha=0}^{n-1} \frac{1}{n} \frac{1 + q^m \omega^\alpha}{(1 - q^m \omega^\alpha)^3} q^m \omega^\alpha = n^2 q^{mn} \frac{1 + q^{mn}}{(1 - q^{mn})^3}, \quad m \neq 0. \tag{2.30}$$

⊙ Note that

$$z\frac{1+z}{(1-z)^3} = \sum_{k=1}^{\infty} k^2 z^k, \quad |z| < 1,$$

hence

$$S_0 = \sum_{k=1}^{\infty} q^{mk} k^2 \frac{1}{n} \sum_{\alpha=0}^{n-1} \omega^{\alpha k}.$$

But

$$\frac{1}{n} \sum_{\alpha=0}^{n-1} \omega^{\alpha k} = \left\{ \begin{array}{ll} 0, & k \neq jn, \quad j \in \mathbb{Z}, \\ 1, & k = jn, \end{array} \right.$$

therefore

$$S_0 = \sum_{j=1}^{\infty} q^{mnj} (nj)^2 = n^2 q^{mn} \frac{1 + q^{mn}}{(1 - q^{mn})^3}. \qquad \blacksquare$$

Using the lemma we find from (2.29):

$$\begin{aligned}
\langle \hat{r}_{0,n}^0 \bar{f}, \bar{g} \rangle &= \frac{n^2}{2} \sum_{m \neq 0} f_m g_{-m} \frac{1 + q^{mn}}{1 - q^{mn}} = \frac{n^2}{2} \left\langle \left[\frac{1 + \hat{h}^n}{1 - \hat{h}^n} (1 - Res) \right] f, g \right\rangle_{\mathbb{C}} \\
&= \left\langle \frac{n}{2} \frac{1 + \hat{h}^n}{1 - \hat{h}^n} \mathcal{P}'_0 \bar{f}, \bar{g} \right\rangle_{L\mathfrak{h}_n},
\end{aligned}$$

as desired. ∎

2.2. **Explicit formula for the quotient bracket.** As mentioned above, the quotient $\mathbb{Y}_n / LN_+ (n)$ can be identified with the space $\mathbb{M}_n$ of scalar q-difference operators of n-th order. To describe the quotient bracket we shall consider $\mathbb{M}_n$ as an affine subspace in the algebra $\Psi \mathbf{D}_q$ of q-pseudodifference symbols. By definition, $\Psi \mathbf{D}_q$ consists of formal series of the form

$$A = \sum_{i=-\infty}^{N(A)} a_i(z) D^i, \quad a_i \in \mathbb{C}\left(\left(z^{-1}\right)\right) \tag{2.31}$$

with the commutation relation

$$D \cdot a = {}^h a \cdot D. \tag{2.32}$$

As a linear space, $\Psi\mathbf{D}_q$ is a direct sum of three subalgebras,

$$J_+ = \left\{ A \in \Psi\mathbf{D}_q | \ A = \sum_{i=1}^{N(A)} a_i\left(z\right) D^i, \quad a_i \in \mathbb{C}\left(\left(z^{-1}\right)\right) \right\}, \tag{2.33}$$

$$J_0 = \mathbb{C}\left(\left(z^{-1}\right)\right) \subset \Psi\mathbf{D}_q, \tag{2.34}$$

$$J_- = \left\{ A \in \Psi\mathbf{D}_q | \ A = \sum_{i=1}^{\infty} a_i\left(z\right) D^{-i}, \quad a_i \in \mathbb{C}\left(\left(z^{-1}\right)\right) \right\}. \tag{2.35}$$

Clearly, J_0 normalizes $J_\pm$ and hence $J_{(\pm)} = J_\pm + J_0$ is also a subalgebra. Let $P_\pm, P_0$ be the associated projection operators which project $\Psi\mathbf{D}_q$ onto $J_\pm, J_0$, respectively, parallel to the complement. Put $P_{(\pm)} = P_\pm + P_0$. For $A \in \Psi\mathbf{D}_q$ set $A_\pm = P_\pm A, \quad A_{(\pm)} = P_{(\pm)}A$.

We define the residue of a q-pseudodifference operator A by

$$\mathrm{Res}A = A_0 = P_0 A.$$

It is easy to see that the formal trace defined by

$$Tr\ A = \int \mathrm{Res}A\,dz/z \tag{2.36}$$

satisfies the natural condition

$$Tr\ AB = Tr\ BA, \quad A, B \in \Psi\mathbf{D}_q.$$

We introduce an inner product in $\Psi\mathbf{D}_q$ by

$$\langle A, B\rangle = Tr\ AB, \quad A, B \in \Psi\mathbf{D}_q. \tag{2.37}$$

Clearly, this inner product is invariant and non-degenerate and the subalgebras $J_\pm$ are isotropic; moreover, it sets J_+ and J_- into duality, while $J_0 \simeq J_0^*$.

We shall now define a class of Poisson brackets on $\Psi\mathbf{D}_q$. The natural algebra of observables $Fun\left(\Psi\mathbf{D}_q\right)$ in the present case is generated by 'elementary' functionals which assign to a pseudodifference operator A the formal integrals of its coefficients,

$$\zeta_i^j\left(A\right) = Tr\left(z^{-j}AD^{-i}\right).$$

As compared to the case of differential operators, the definition of a quadratic Poisson bracket in the difference case is not quite straightforward; the point is that the 'naive' bracket defined by analogy with the differential case is not compatible with the natural normalization condition for difference operators (highest coefficient is set to one); an easy scrutiny shows that the source of the trouble lies in the J_0-component in the expansion

$$\Psi\mathbf{D}_q = J_+ \dot{+} J_0 \dot{+} J_-.$$

To avoid this difficulty we are bound to consider a more general class of quadratic Poisson bracket which mix together both left and right gradients of functions[1]. (Recall that in the Gelfand-Dickey case left and right gradients are coupled only to the gradients of the same chirality.)

For a smooth functional φ let us write

$$D\varphi = \left(\begin{array}{c} \nabla\varphi \\ \nabla'\varphi \end{array} \right).$$

Let us consider quadratic Poisson brackets on $\Psi \mathbf{D}_q$ of the following form:

$$\{\varphi, \psi\} = \left\langle \left\langle \left(\begin{array}{cc} R + aP_0 & bP_0 \\ cP_0 & R + dP_0 \end{array} \right) D\varphi, D\psi \right\rangle \right\rangle, \qquad (2.38)$$

where $R = \frac{1}{2}(P_+ - P_-)$ and a, b, c, d are linear operators acting in J_0 satisfying

$$a = -a^*, \quad d = -d^*, \quad c^* = b.$$

In other words, the bracket (2.38) differs from the naive Gelfand-Dickey bracket by a 'perturbation term' which is acting only on the J_0-components of the gradients. This bracket satisfies the Jacobi identity for any choice of a, b, c, d. Note that different a, b, c, d may give rise to the same bracket. More precisely, we have the following

Lemma 2.8. *Let f, g, h, k be linear operators in J_0 with images in the subspace of constants $\mathbb{C} \cdot 1 \subset J_0$. The r-matrices $\mathcal{R}$ and $\mathcal{R}' = \mathcal{R} + \Theta$ where*

$$\Theta = \left(\begin{array}{cc} h - k^* & f + k^* \\ h + g^* & -g^* + f \end{array} \right),$$

define the same Poisson bracket.

Up to this ambiguity, the unique choice of the coefficients a, b, c, d is assured by the condition that the set $\mathbb{M}_n$ of difference operators with normalized highest coefficient is a Poisson submanifold with respect to the Poisson structure (2.38) and that, moreover, formal spectral invariants of difference operators give rise to Lax equations of standard commutator form. More precisely, we have the following theorem (see [14]):

Theorem 2.9. *There exists a unique Poisson bracket of the form (2.38) on $\Psi \mathbf{D}_q$ such that*

1) *the affine subspace $\mathbb{M}_n$ is a Poisson submanifold;*
2) *Formal spectral invariants $H_m = \frac{n}{m} Tr\, L^{\frac{m}{n}}, \quad m \in \mathbb{N},$ are in involution.*

---

[1]This class of Poisson brackets naturally arises in the theory of Poisson Lie groups, cf. [16], [5], [12].

This bracket is given by [2]

$$\{\varphi,\psi\}_n^0 = \left\langle\left\langle\left(\begin{array}{cc} R + \left(\frac{1}{2}\frac{1+\hat{h}^n}{1-\hat{h}^n}\right)P_0' & -\left(\frac{\hat{h}^n}{1-\hat{h}^n}\right)P_0' \\ \left(\frac{1}{1-\hat{h}^n}\right)P_0' & R - \left(\frac{1}{2}\frac{1+\hat{h}^n}{1-\hat{h}^n}\right)P_0' \end{array}\right)D\varphi, D\psi\right\rangle\right\rangle. \tag{2.39}$$

Remark 2.4. *It is easy to see that the involutivity condition (2) is equivalent to the following simple constraint:*

$$a + b = c + d. \tag{2.40}$$

The quotient bracket on the set of q-difference operators which is obtained via the q-DS reduction differs from the above formula by an additional term which reflects the remaining freedom in the choice of the classical r-matrix on $L\mathfrak{gl}_n$ which is compatible with the reduction; namely:

Theorem 2.10. *Let*

$$\hat{r}_{\Delta,n}^0 = \frac{1}{2}\frac{1 + \hat{h}\hat{\tau}_n}{1 - \hat{h}\hat{\tau}_n}\mathcal{P}_0' + n\Delta\mathcal{P}_{U_n}. \tag{2.41}$$

where $\mathcal{P}_{U_n}$ is the orthogonal projection operator onto U_n and Δ is a skew symmetric operator in U_n commuting with $\hat{h}$. Let $\hat{r}_{\Delta,n} = \frac{1}{2}(\mathcal{P}_+ - \mathcal{P}_-) + \hat{r}_{\Delta,n}^0\mathcal{P}_0$.
The Poisson bracket $\{\cdot,\cdot\}_{\Delta,n}$ on $L\mathfrak{gl}_n$ defined by

$$\left\{\hat{\varphi},\hat{\psi}\right\}_{\Delta,n} = \left\langle\left\langle\left(\begin{array}{cc} \hat{r}_{\Delta,n} & -\hat{h}\left(\hat{r}_{\Delta,n}\right)_+ \\ \left(\hat{r}_{\Delta,n}\right)_-\hat{h}^{-1} & -\hat{r}_{\Delta,n} \end{array}\right)\left(\begin{array}{c}\nabla\hat{\varphi}\\\nabla'\hat{\varphi}\end{array}\right), \left(\begin{array}{c}\nabla\hat{\psi}\\\nabla'\hat{\psi}\end{array}\right)\right\rangle\right\rangle \tag{2.42}$$

gives rise via DS-reduction to the following bracket on $\mathbb{M}_n$:

$$\{\varphi,\psi\}_n^\Delta = \left\langle\left\langle\left(\begin{array}{cc} R + \left(\frac{1}{2}\frac{1+\hat{h}^n}{1-\hat{h}^n} + \Delta\right)P_0' & -\left(\frac{1}{1-\hat{h}^n} + \Delta\right)\hat{h}^n P_0' \\ \left(\frac{\hat{h}^n}{1-\hat{h}^n} + \Delta\right)\hat{h}^{-n}P_0' & R - \left(\frac{1}{2}\frac{1+\hat{h}^n}{1-\hat{h}^n} + \Delta\right)P_0' \end{array}\right)D\varphi, D\psi\right\rangle\right\rangle; \tag{2.43}$$

(here we have identified $EndU_n$ and $End\mathbb{C}((z^{-1}))$.)

The remaining ambiguity in the choice of r-matrix may be removed if we impose the involutivity condition.

Theorem 2.11. *The only one of brackets $\{\cdot,\cdot\}_{\Delta,n}$ which gives rise to a Poisson bracket on $\mathbb{M}_n$ satisfying the involutivity condition (2.40) is $\{\cdot,\cdot\}_{0,n}$.*

Proof. In the class of the brackets (2.43) only the bracket $\{\cdot,\cdot\}_n^0$ satisfies this condition. Indeed, (2.40) implies that $\Delta\left(2 - \hat{h}^n - \hat{h}^{-n}\right) = 0$ but the operator $2 - \hat{h}^n - \hat{h}^{-n}$ is invertible.∎

---

[2]For difference Lax equations on the lattice Poisson bracket 2.39 was also introduced in [13].

Proof of theorem 2.10. For $\Delta = 0$ this theorem has been proved in [14], hence we need to calculate only the contribution of the term $n\Delta\mathcal{P}_{U_n}$. Let φ be a smooth functional on $\mathbb{M}_n$, $\hat{\varphi}$ the corresponding $LN_+(n)$-invariant functional on $L\mathfrak{gl}_n$. To fix their gradients we shall assume that

$$d\varphi = \sum_{i=0}^{n-1} f_i D^{-i}, \quad f_i \in \mathbb{C}((z^{-1})), \tag{2.44}$$

and $d\hat{\varphi} \in L\mathfrak{b}_-(n)$. For $L \in \mathbb{M}_n$ let us denote by $\mathbb{L} \in L\mathfrak{gl}_n$ the corresponding companion matrix.

Let us denote by $J_\Delta\left(\hat{\varphi}, \hat{\psi}\right)$ the contribution of $n\Delta\mathcal{P}_{U_n}$ to the bracket (2.42). From (2.21) it follows that

$$J_\Delta\left(\hat{\varphi}, \hat{\psi}\right) = \left\langle n\Delta\mathcal{P}_{U_n} Z_{\hat{\varphi}}^0, Z_{\hat{\psi}}^0 \right\rangle, \tag{2.45}$$

where $Z_{\hat{\varphi}}^0 \equiv \mathcal{P}_0 Z_{\hat{\varphi}}, \quad Z_{\hat{\psi}}^0 \equiv \mathcal{P}_0 Z_{\hat{\psi}}$.

Lemma 2.12. *We have*

$$Z_{\hat{\varphi}}^0(\mathbb{L}) = \text{diag}\left(h^{-1} P_0 \nabla\varphi(L), 0, \ldots, 0, -P_0 \nabla'\varphi(L)\right). \tag{2.46}$$

Lemma 2.13. *The projection operator $\mathcal{P}_{U_n}$ is given by*

$$\mathcal{P}_{U_n} \cdot \text{diag}\left(F_0(z), \ldots F_{n-1}(z)\right) = \text{diag}\left(f_0(z), f_0\left(q^{-1}z\right), \ldots, f_0\left(q^{-(n-1)}z\right)\right), \tag{2.47}$$

where

$$f_0(z) = \frac{1}{n} \sum_{i=0}^{n-1} F_i\left(q^i z\right). \tag{2.48}$$

Using these lemmas we find

$$\mathcal{P}_{U_n} Z_{\hat{\varphi}}^0 = \frac{1}{n}\left(h^{-1} P_0 \nabla\varphi - h^{n-1} P_0 \nabla'\varphi\right).$$

Substituting this into (2.45) and taking into account the invariance of the inner product we obtain

$$J_\Delta\left(\hat{\varphi}, \hat{\psi}\right) = \left\langle\!\!\left\langle\left(\begin{matrix} \Delta P_0 & -\Delta\hat{h}^n P_0 \\ \Delta\hat{h}^{-n} P_0 & -\Delta P_0 \end{matrix}\right) D\varphi, D\psi\right\rangle\!\!\right\rangle. \tag{2.49}$$

But Δ is skew-symmetric, hence it annihilates the one-dimensional subspace $\mathbb{C} \cdot 1 \subset \mathbb{C}\left((z^{-1})\right)$ and we may replace P_0 by P_0' in (2.49). ∎

3. Algebras $\mathfrak{gl}_q^\lambda$ of complex size matrices and their loop algebras.

In this section we construct an algebra $\mathfrak{gl}_q$ consisting of $\mathfrak{gl}_\infty$-matrices whose matrix elements are holomorphic functions of special form. Then we define a trace functional on $\mathfrak{gl}_q$ with values in the space $Hol\,(\mathbb{C})$ of holomorphic functions; it satisfies the natural condition $Tr\,AB = Tr\,BA$.

For any fixed $\lambda \in \mathbb{C}$ the algebra $\mathfrak{gl}_q^\lambda \subset \mathfrak{gl}_\infty$ is the image of $\mathfrak{gl}_q$ under the evaluation map $A \mapsto A\,(\lambda)$. The functional Tr on $\mathfrak{gl}_q$ induces a $\mathbb{C}$-valued trace on $\mathfrak{gl}_q^\lambda$. This construction is a q-difference counterpart of the one described in [11]; in particular, $\mathfrak{gl}_q$ and $\mathfrak{gl}_q^\lambda$ are some extensions of the algebras $\mathfrak{gl}, \overline{\mathfrak{gl}}_\lambda$ considered there. At the end of this section we shall describe the loop algebras $L\mathfrak{gl}_q$, $L\mathfrak{gl}_q^\lambda$.

3.1. $\mathcal{A}_0$-**functions.** We shall describe a class $\mathcal{A}_0 \subset Hol\,(\mathbb{C})$ of holomorphic functions we shall deal with throughout this article.

By definition, $\mathcal{A}_0$ is the algebra of functions of complex variable w generated by w, q^w, q^{-w}, where $q^w \equiv \exp\,(w \ln q)$. The elements of $\mathcal{A}_0$ will be called $\mathcal{A}_0$-functions. Evidently, the set of elements $\zeta_{m,n} = w^m q^{nw}$, $m \in \mathbb{Z}_+$, $n \in \mathbb{Z}$, is a linear basis of $\mathcal{A}_0$, i.e. any $\mathcal{A}_0$-function f can be decomposed into a *finite sum* with respect to this basis:

$$f\,(w) = \sum_{m+|n| \le N(f)} f_{m,n} \zeta_{m,n}, \quad f_{m,n} \in \mathbb{C}. \tag{3.1}$$

The minimal possible value of $N(f)$ in the sum (3.1) is called the degree of f and will be denoted by $\deg f$. Note also that the set of subspaces $\mathbb{C}\zeta_{m,n}$ defines a $\mathbb{Z}_+ \times \mathbb{Z}$-grading on $\mathcal{A}_0$.

$\mathcal{A}_0$-functions satisfy two important properties which will be widely used below. The first one called *interpolation property* allows to reconstruct an $\mathcal{A}_0$-functions from its values at sufficiently large integer points:

Proposition 3.1. *Let* $f \in \mathcal{A}_0$ *and* $f\,(n) = 0$ *for all sufficiently large integer* n, *then* $f\,(w) \equiv 0$.

Hence if some relation for $\mathcal{A}_0$-functions holds for sufficiently large integer values of w, it holds also for all $w \in \mathbb{C}$.

The second property is given by

Proposition 3.2. *For any* $\mathcal{A}_0$-*function* f *and any* $l \in \mathbb{Z}$ *there exists a unique* $\mathcal{A}_0$-*function* $\tilde{F}$ *which interpolates the sum* $F\,(n) = \sum_{i=0}^{n-1} f\,(i)\,q^{il}$, *i.e.,* $\tilde{F}\,(n) = F\,(n)$, $n \in \mathbb{N}$.

It will play the key role in the definition of trace as well as in the proof of the cross-section theorem, see below.

We say that a function $f(w,t)$ is an $\mathcal{A}_0$-function of two complex variables w, t if it can be written as a finite sum

$$f(w,t) = \sum_{i=1}^{N(f)} f_i^{(1)}(w) \cdot f_i^{(2)}(t), \quad f_i^{(1)}, f_i^{(2)} \in \mathcal{A}_0.$$

In other words, the space of $\mathcal{A}_0$-function of two variables is the algebraic tensor product $\mathcal{A}_0 \otimes \mathcal{A}_0$.

Proposition 3.3. *Let $f(w,t)$ be an $\mathcal{A}_0$-function of variable w for any fixed t and a $\mathcal{A}_0$-function of variable t for any fixed w, then it is an $\mathcal{A}_0$-function of two variables.*

3.2. **Algebras $\mathfrak{gl}_q$, $\mathfrak{gl}_q^\lambda$ and trace functional.** Let $\mathfrak{a}$ be an associative algebra. We define $\mathfrak{gl}_\infty(\mathfrak{a})$ as the algebra of semi-infinite matrices $A = \{A_{i,j} \in \mathfrak{a}\}_{i,j=0,1,\dots}$, such that $A_{i,j} = 0$ if $i - j > N(A)$. For $\mathfrak{gl}_\infty(\mathbb{C})$ we write simply $\mathfrak{gl}_\infty$. Note that if $\mathfrak{a}$ is infinite dimensional, the algebra $\mathfrak{gl}_\infty(\mathfrak{a})$ is wider than the algebraic tensor product $\mathfrak{gl}_\infty \otimes \mathfrak{a}$.

Definition 3.1. *The algebra $\mathfrak{gl}_q \subset \mathfrak{gl}_\infty(\mathcal{A}_0)$ consists of $\mathfrak{gl}_\infty$-matrices $A(t) = \{A_{i,j}(t)\}$ with coefficients in $\mathcal{A}_0$ satisfying the following conditions:*
   There exists an integer $N(A)$ such that

   1. $A_{i,i+n}(t) = 0$ *if* $n < -N(A)$;
   2. *for any fixed* $n \geq -N(A)$ *and any* $i > N(A)$ $A_{i,i+n}(t)$ *considered as a function of variables* i, t *can be interpolated by an $\mathcal{A}_0$-function of two variables;*
   3. *For all integer* $m > N(A)$ *and* $N(A) < i < m$, $j \geq m$ *we have* $A_{i,j}(m) = 0$.

   The minimal possible value of $N(A)$ is called the regularity degree of A *and will be denoted by* $\mathrm{reg}A$

   In other words, condition 2 means that
   i) for any fixed t $A_{i,i+n}(t)$ considered as a function of i can be interpolated by an $\mathcal{A}_0$-function;
   ii) the degree of $A_{i,i+n}(t)$ considered as an $\mathcal{A}_r$-function of t is uniformly bounded for all i.
   Condition 3 means that the matrix $A(m) \subset \mathfrak{gl}_\infty$ has the form:

$$A(m) = \left(\begin{array}{c|c} a \in \mathfrak{gl}_m & b \\ \hline * & * \end{array} \right),$$

where the number of non-zero rows in the right upper block b does not exceed $\mathrm{reg}A$ and hence is uniformly bounded for all m.

We define the following $\mathbb{Z}$-grading on $\mathfrak{gl}_q$: the set of elements of level n consists of matrices with only n-th non-zero diagonal, i.e. $A_{i,i+k}(t) = 0$ if $k \neq n$. We will denote by $A^{(n)}$ the $\mathcal{S}_n$-component of a matrix $A \in \mathfrak{gl}_q$.

For any matrix $A \in \mathcal{S}_n$ we can assign a $\mathcal{A}_0$-function of two variables. By definition of $\mathfrak{gl}_q$, there exists an $\mathcal{A}_0$-function $f(w,t)$ which interpolates $A_{i,i+n}(t)$ *for all sufficiently large i*:

$$A_{i,i+n}(t) = f(i,t), \quad \forall i > N(A), \quad \forall t \in \mathbb{C}.$$

We will denote it by $A(w,t)$.

So, for $A \in \mathfrak{gl}_q$ its n-th diagonal $A^{(n)} \in \mathcal{S}_n$ and $A^{(n)}(w,t)$ is the corresponding $\mathcal{A}_0$-function. For positive integer n these functions satisfy the following important property which ensures the invariance of the trace functional on $\mathfrak{gl}_q$:

Proposition 3.4. *For any $A \in \mathfrak{gl}_q$, $n \in \mathbb{N}$, we have:*

$$A^{(n)}(w, w+l) = 0, \quad \forall l = 1, \ldots, n, \quad \forall w \in \mathbb{C}. \tag{3.2}$$

Proof. By definition of $\mathfrak{gl}_q$, $A_{i,i+n}(m) = 0$ if $m > N(A)$, $N(A) < i < m$, and $i + n \geq m$, or, equivalently, if $i = m - l$, $\forall l = 1, \ldots, n$. But for any $i > N(A)$ $A_{i,i+n}(m)$ coincides with its interpolating $\mathcal{A}_0$-function. Hence $A^{(n)}(m - l, m) = 0$ for any integer $m > N(A) + l$, and therefore by proposition 3.1 $A^{(n)}(w - l, w) = 0$ for $\forall w \in \mathbb{C}$, which is equivalent to (3.2). ∎

Now we shall define the trace functional on $\mathfrak{gl}_q$. This construction is parallel to the one described by Khesin and Malikov in [11] and goes back to J.Bernstein.

Let us consider the sum

$$F_A(n,t) = \sum_{i=0}^{n-1} A_{i,i}(t). \tag{3.3}$$

By proposition 3.2, there exists a unique $\mathcal{A}_0$-function of two variables which coincides with $F_A(n,t)$ for any sufficiently large integer n. We will denote it by $\mathbb{D}_A(w,t)$. By definition,

$$(Tr\, A)(t) = \mathbb{D}_A(t,t) \in \mathcal{A}_0. \tag{3.4}$$

Proposition 3.5.

$$Tr\, AB = Tr\, BA.$$

Proof. It is evident that Tr is consistent with the grading, i.e. for any elements $A \in \mathcal{S}_i$, $B \in \mathcal{S}_j$ the trace of their product vanishes unless $i + j = 0$. Hence it is sufficient to consider the case of $A \in \mathcal{S}_k$, $B \in \mathcal{S}_{-k}$, $k \in \mathbb{N}$. For all sufficiently

large $n \in \mathbb{N}$ we have

$$
\mathbb{D}_{AB}(n,t) - \mathbb{D}_{BA}(n,t) = F_{AB}(n,t) - F_{AB}(n,t) = \sum_{i=n-k}^{n-1} A(i,t)B(i+k,t)
$$

$$
= \sum_{j=1}^{k} A(n-j,t)B(n-j+k,t).
$$

All terms of this expression are $\mathcal{A}_0$-functions of variable n, therefore, by the interpolation property, it holds for all complex values of n; in particular, for $n = t$ we obtain

$$
Tr\,AB - Tr\,BA \equiv \mathbb{D}_{AB}(t,t) - \mathbb{D}_{BA}(t,t) = \sum_{j=1}^{k} A(t-j,t)B(t-j+k,t).
$$

But $A(t-j,t) = 0$ for $j = 1, \ldots, k$, by proposition 3.4. $\blacksquare$

We use the following notation: $\mathfrak{b}_+(\mathfrak{b}_-) \subset \mathfrak{gl}_q$ is the subalgebra of upper (lower) triangular matrices, $\mathfrak{n}_+(\mathfrak{n}_-)$ are the corresponding subalgebras of strictly triangular matrices and $\mathfrak{h}$ is the subalgebra of diagonal matrices. The set $\mathfrak{N}_+ \subset \mathfrak{gl}_q$ of matrices of the form $T = 1 + S$, $S \in \mathfrak{n}_+$, is an infinite-dimensional Lie group with Lie algebra $\mathfrak{n}_+$.

Let us fix $\lambda \in \mathbb{C}$ and consider the evaluation map:

$$
i_\lambda : \mathfrak{gl}_\infty(\mathcal{A}_0) \to \mathfrak{gl}_\infty, \quad A \mapsto A(\lambda). \tag{3.5}
$$

We shall use the following notation: for a subset K of $\mathfrak{gl}_q$ we denote by K^λ its image under the evaluation map (3.5).

Our main object, the algebra $\mathfrak{gl}_q^\lambda \subset \mathfrak{gl}_\infty$ is the image of the whole $\mathfrak{gl}_q$. We define also its subalgebras $\mathfrak{b}_\pm^\lambda, \mathfrak{n}_\pm^\lambda, \mathfrak{h}^\lambda$ and the group $\mathfrak{N}_+^\lambda$.

The algebra $\mathfrak{gl}_q^\lambda$ is $\mathbb{Z}$-graded with respect to the set of its subspaces $\mathcal{S}_n^\lambda$, $n \in \mathbb{Z}$. For any $n \in \mathbb{N}$ $\mathfrak{gl}_n$ is naturally embedded into $\mathfrak{gl}_q^\lambda$ as its left upper block:

$$
\begin{pmatrix} \mathfrak{gl}_n & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.6}
$$

For a matrix $A \in \mathfrak{gl}_q^\lambda$ we denote this upper block by $A_{|n}$.

The $\mathcal{A}_0$-valued trace functional on $\mathfrak{gl}_q$ induces the ordinary $\mathbb{C}$-valued trace on $\mathfrak{gl}_q^\lambda$: we must put $t = \lambda$ in (3.4); it will be denoted by the same symbol. The restriction of Tr to $\mathfrak{gl}_n \subset \mathfrak{gl}_q^\lambda$ coincides with the standard matrix trace. The corresponding invariant inner product on $\mathfrak{gl}_q^\lambda$ is non-degenerate: indeed, a matrix which is orthogonal to the all $\mathfrak{gl}_n$, $n \in \mathbb{N}$, is zero.

3.3. **Loop algebras** $L\mathfrak{gl}_q$, $L\mathfrak{gl}_q^\lambda$. Due to infinite dimension of $\mathfrak{gl}_q$, $\mathfrak{gl}_q^\lambda$, an accurate definition of its loop algebras requires some work. The definitions below have the aim to ensure

(1) the existence of a generalized trace functional and of the corresponding invariant inner product on $L\mathfrak{gl}_q$, $L\mathfrak{gl}_q^\lambda$;

(2) the possibility to generalize the cross-section theorem 2.1 to the cases of $L\mathfrak{gl}_q$, $L\mathfrak{gl}_q^\lambda$.

Let $\mathfrak{a}$ be an associative algebra; we shall denote by $\mathfrak{a}((z^{-1}))$ the space of formal Laurent series with coefficients in $\mathfrak{a}$. For $A \in \mathfrak{a}((z^{-1}))$ we denote by $A^{[m]}$ its Laurent coefficient corresponding to z^m.

Let us consider the algebra $\mathfrak{gl}_\infty(\mathcal{A}_0((z^{-1})))$, i.e. the algebra of $\mathfrak{gl}_\infty$-matrices A whose matrix coefficients $A_{ij}(t,z)$ are formal Laurent series in z with co-efficients in $\mathcal{A}_0$. For any $m \in \mathbb{Z}$ Laurent coefficients $A_{ij}^{[m]}(t)$ form a matrix $A^{[m]} = \{A_{ij}^{[m]}(t)\} \in \mathfrak{gl}_\infty(\mathcal{A}_0)$. Note that in general $A^{[m]} \neq 0$ for all $m \in \mathbb{Z}$, however, for any fixed i,j there exists an integer $\tilde{N}(A,i,j)$ such that $A_{ij}^{[m]}(t) = 0$ if $m > \tilde{N}(A,i,j)$.

Definition 3.2. *Loop algebra $L\mathfrak{gl}_q$ consists of matrices $A \in \mathfrak{gl}_\infty(\mathcal{A}_0((z^{-1})))$ satisfying the following conditions:*

(1) for any $m \in \mathbb{Z}$ $A^{[m]} \in \mathfrak{gl}_q$;

(2) there exists an integer $N(A)$ such that $\mathrm{reg}A^{[m]} \leq N(A)$;

(3) for any $n \in \mathbb{Z}$ there exists an integer $\tilde{N}(A,n)$ such that $A_{i,i+n}^{[m]}(t) = 0$ if $m > \tilde{N}(A,n)$.

The notion of regularity degree can be naturally generalized to the case of the loop algebra $L\mathfrak{gl}_q$, i.e., the regularity degree $\mathrm{reg}A$ of a matrix $A \in L\mathfrak{gl}_q$ is the minimal possible value of $N(A)$.

We define the $\mathcal{A}_0((z^{-1}))$-valued trace functional on $L\mathfrak{gl}_q$ by the same formula as above. In a similar way we may prove that the trace satisfies $Tr\,AB = Tr\,BA$.

The diagonal grading $\{\mathcal{S}_i\}$, the evaluation map (3.5), the definitions of subalgebras $\mathfrak{b}_\pm^\lambda, \mathfrak{n}_\pm^\lambda, \mathfrak{h}^\lambda$ and the group $\mathfrak{N}_+^\lambda$ have their natural counterparts in the case of $L\mathfrak{gl}_q$. For a subset $K \subset \mathfrak{gl}_q\left(K^\lambda \subset \mathfrak{gl}_q^\lambda\right)$ we shall denote by $LK\left(LK^\lambda\right)$ the corresponding subset in $L\mathfrak{gl}_q\left(L\mathfrak{gl}_q^\lambda\right)$.

4. Gauge orbits of the upper triangular group and the cross-section theorem.

In this section we define the gauge action of the upper triangular group and describe a cross-section of this action. We consider the case of $\mathfrak{gl}_q$, the corresponding assertion for $\mathfrak{gl}_q^\lambda$ may be obtained by application of the evaluation map.

Let us denote by $\mathbb{Y}_q \subset L\mathfrak{gl}_q$ the affine subspace of matrices of the form $\mathcal{L} = \Lambda + A$, where $A \in L\mathfrak{b}_+$ and

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & \ldots \\ 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \ddots & \ldots \end{pmatrix}. \tag{4.1}$$

We define the gauge action of $L\mathfrak{N}_+$ by

$$\mathcal{L} \mapsto {}^h T \cdot \mathcal{L} \cdot T^{-1}, \quad T \in L\mathfrak{N}_+. \tag{4.2}$$

Evidently, the space $\mathbb{Y}_q$ is preserved by this action.

Theorem 4.1.

1. *The gauge action of $L\mathfrak{N}_+$ on $\mathbb{Y}_q$ is free.*
2. *The set of companion matrices, i.e. matrices of the form*

$$\widetilde{\mathcal{L}} = \begin{pmatrix} u_1(t,z) & u_2(t,z) & u_3(t,z) & \ldots \\ 1 & 0 & 0 & \ldots \\ 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \ddots & \ldots \end{pmatrix}, \quad u_i(t,z) \in \mathcal{A}_0((z^{-1})), \tag{4.3}$$

   is a cross-section of this action.

Proof. Let $T \in L\mathfrak{N}_+$ be an element which converts $\mathcal{L} \in \mathbb{Y}_q$ into a companion matrix $\widetilde{\mathcal{L}}$. The statements of the theorem mean that the equation

$$ {}^h T \cdot \mathcal{L} = \widetilde{\mathcal{L}} T \tag{4.4}$$

has a unique solution. Let us write $T, \mathcal{L}, \widetilde{\mathcal{L}}$ in the form:

$$\mathcal{L} = \Lambda + \sum_{i \geq 0} \mathcal{L}^{(i)}, \quad \widetilde{\mathcal{L}} = \Lambda + \sum_{i \geq 0} \widetilde{\mathcal{L}}^{(i)}, \quad T = 1 + \sum_{j > 0} T^{(j)},$$

where the superscripts $^{(i)}$ denote as above the i-th diagonal component of the corresponding matrices. Substituting this into (4.4) we obtain the following infinite sequence of equations:

$$\begin{cases} {}^h T^{(1)}\Lambda - \Lambda T^{(1)} = -\mathcal{L}^{(0)} + \widetilde{\mathcal{L}}^{(0)}, \\ {}^h T^{(i)}\Lambda - \Lambda T^{(i)} = -\mathcal{L}^{(i-1)} + \widetilde{\mathcal{L}}^{(i-1)} + \sum_{j=1}^{i-1} \left(\widetilde{\mathcal{L}}^{(i-j-1)} T^{(j)} - {}^h T^{(j)} \mathcal{L}^{(i-j-1)} \right), \ i \geq 2. \end{cases} \tag{4.5}$$

The i-th equation in this sequence is an equation for $T^{(i)}$ and u_i. We must prove that:

1) for all $i \in \mathbb{N}$ the corresponding equation has an unique solution $T^{(i)} \in L\mathcal{S}_i$, $u_i \in \mathcal{A}_0((z^{-1}))$;

2) there exists an integer $N(T)$ such that $\mathrm{reg}T^{(i)} \leq N(T)$ for all $i \in \mathbb{N}$.

The last condition allows to combine all $T^{(i)}$ into a single matrix $T \in L\mathfrak{N}_+$, the regularity degree of T being equal or less than $N(T)$. We shall prove not only that $N(T)$ exists, but also that we may put $N(T) = \mathrm{reg}\mathcal{L}$.

We shall prove assertion 1), 2) inductively.

Let us rewrite the first equation of (4.5) as follows:

$$\begin{cases} T_{0,1}(t, qz) = -\mathcal{L}_{0,0}(t, z) + u_1(t, z), \\ T_{n,n+1}(t, qz) - T_{n-1,n}(t, z) = -\mathcal{L}_{n,n}(t, z), \quad n \geq 1. \end{cases} \tag{4.6}$$

The base of induction is the following

Lemma 4.2.

1) The equation (4.6) has a unique solution $T^{(1)} \in LS_1$, $u_1 \in \mathcal{A}_0((z^{-1}))$.

2) $\mathrm{reg}T^{(1)} \leq \mathrm{reg}\mathcal{L}$.

$\odot$ From (4.6) it follows that

$$T_{n,n+1}(t, z) = -\sum_{i=0}^{n} \mathcal{L}_{i,i}(t, q^{i-n-1}z) + u_1(t, q^{-n-1}z). \tag{4.7}$$

$T^{(1)} \in LS_1$ implies that there exists a $\mathcal{A}_0$-function $T^{(1)}(w, t, z)$ which interpolates $T_{n,n+1}(t, z)$ for sufficiently large integer $n > N_1$:

$$T_{n,n+1}(t, z) = T^{(1)}(n, t, z). \tag{4.8}$$

By proposition 3.4,

$$T^{(1)}(n, n+1, z) = 0. \tag{4.9}$$

Substituting (4.8) and (4.9) into (4.7) we find

$$u_1(n+1, z) = \sum_{i=0}^{n} \mathcal{L}_{i,i}(n+1, q^i z), \quad \forall n > N_1. \tag{4.10}$$

There exists a unique function $u_1 \in \mathcal{A}_0((z^{-1}))$ satisfying (4.10). Indeed, developing (4.10) in powers of z we obtain the following relation for the coefficients $u_1^m(t) \in \mathcal{A}_0$:

$$z^m: \quad u_1^m(n+1) = \sum_{i=0}^{n} \mathcal{L}_{i,i}^m(n+1) \cdot q^{im}, \quad \forall n > N_1. \tag{4.11}$$

By definition of regularity degree, for $i \geq \mathrm{reg}\mathcal{L}$ all coefficients $\mathcal{L}_{i,i}^m(t)$ can be interpolated by $\mathcal{A}_0$-functions of two variables i, t. Therefore, by proposition 3.2, the whole sum in the r.h.s. of (4.11) also may be interpolated by a (unique) $\mathcal{A}_0$-function for $\forall n > \mathrm{reg}\mathcal{L}$, and we may put $N_1 = \mathrm{reg}\mathcal{L}$.

Once u_1 is known, $T^{(1)}$ is uniquely defined by the equation (4.7) ; we can verify in the same way as above that $\mathrm{reg}T^{(1)} \leq \mathrm{reg}\mathcal{L}$. $\blacksquare$

Assume now that the first l equations in (4.5) have unique solutions $T^{(i)} \in LS_i$, $u_i \in \mathcal{A}_0((z^{-1}))$, $i = 1, \ldots, l$, and that $\mathrm{reg}T^{(i)} \leq \mathrm{reg}\mathcal{L}$.

The $(l+1)$-th equation has the form:

$$T_{n,n+l+1}(t, qz) - T_{n-1,n+l}(t, z) = \delta_{n0} u_{l+1} - F_{n,n+l}^{(l)}(t, z), \qquad (4.12)$$

where δ is the Kronecker symbol and $F^{(l)} \in LS_l$ is defined by

$$F^{(l)} = \mathcal{L}^{(l)} - \sum_{j=1}^{l} \left(\widetilde{\mathcal{L}}^{(l-j)} T^{(j)} -^h T^{(j)} \mathcal{L}^{(l-j)} \right).$$

It is easy to see that $\mathrm{reg} F^{(l)} \le \mathrm{reg} \mathcal{L}$.

The condition $T^{(l+1)} \in LS_{l+1}$ imposes $l+1$ restrictions on $T^{(l+1)}$:

$$T^{(l+1)}(n, n+k, z) = 0, \quad k = 1, \dots l+1. \qquad (4.13)$$

In the same way as above the equation (4.13), corresponding to $k = l+1$, uniquely defines the coefficient u_{l+1}:

$$u_{l+1}(n+l+1, z) = \sum_{i=0}^{n} F_{i,i+l}^{(l)}\left(n+l+1, q^i z\right), \quad \forall n > \mathrm{reg} \mathcal{L}. \qquad (4.14)$$

Then we find $T^{(l+1)}$ from

$$T_{n,n+l+1}(t, z) = -\sum_{i=0}^{n} F_{i,i+l}^{(l)}\left(t, q^{i-n-1} z\right) + u_{l+1}\left(t, q^{-n-1} z\right). \qquad (4.15)$$

$\mathrm{reg} F^{(l)} \le \mathrm{reg} \mathcal{L}$ implies that $T_{n,n+l+1}$ can be interpolated by a $\mathcal{A}_0$-function for $n > \mathrm{reg} \mathcal{L}$.

It remains to verify that the conditions (4.13) for $1 \le k \le l$ are also satisfied. Fix some k. Note that $T^{(l+1)}(n, n+l+1, z) = 0$ for any $n > \mathrm{reg} \mathcal{L}$, therefore, from (4.12) we find:

$$T^{(l+1)}\left(n+l+1-k, n+l+1, q^{l+1-k} z\right) = -\sum_{i=1}^{l+1-k} F^{(l)}\left(n+i, n+l+1, q^{i-1} z\right).$$

All terms in the r.h.s. are zero; indeed, by construction, $F^{(l)} \in LS_l$ and hence $F^{(l)}(w, w+j, z) = 0$ for all $w \in \mathbb{C}$ and for all j satisfying $1 \le j \le l$. Put $w = n+i, j+l+1-i$; clearly, j lies in the prescribed range. So, $T^{(l+1)}(n, n+k, z) = 0$ for any $n > \mathrm{reg} \mathcal{L} + l + 1$. The interpolation property gives $T^{(l+1)}(w, w+k, z) = 0$ for any $w \in \mathbb{C}$. But $T_{n,n+l+1}(w, z) = T^{(l+1)}(n, w, z)$, for all $n > \mathrm{reg} \mathcal{L}$, $w \in \mathbb{C}$ and hence $T_{n,n+l+1}(n+k, z) = 0$ for any $n > \mathrm{reg} \mathcal{L}$, as desired. ∎

5. The choice of r-matrix.

Let us fix $\lambda \in \mathbb{C}$. Like in the finite-dimensional case in order to define the generalized DS-reduction we need to find a Poisson bracket on $L\mathfrak{gl}_q^\lambda$ satisfying the following conditions:

1) the gauge action of $L\mathfrak{N}_+^\lambda$ is admissible;

2) the Poisson bracket of any $L\mathfrak{N}_+^\lambda$ -invariant function $\widehat{\psi}$, $\widehat{\psi}_{|\mathbb{Y}_q^\lambda} = \text{const}$, with arbitrary $L\mathfrak{N}_+^\lambda$-invariant function vanishes on $\mathbb{Y}_q^\lambda$.

We shall use the notation similar to the one of the section 2: $\mathcal{P}_+, \mathcal{P}_-, \mathcal{P}_0$ are the projection operators onto $L\mathfrak{n}_+^\lambda$, $L\mathfrak{n}_-^\lambda$, $L\mathfrak{h}^\lambda$, respectively; $r = \frac{1}{2}(\mathcal{P}_+ - \mathcal{P}_-) + r_0\mathcal{P}_0$, $r_0 \in \text{End}\left(L\mathfrak{h}^\lambda\right)$; $r_\pm = r \pm \frac{1}{2}$. The invariant product on $L\mathfrak{gl}_q^\lambda$ is defined by

$$\langle A(z), B(z)\rangle = \int \frac{dz}{z} Tr\, A(z) B(z).$$

As in the finite-dimensional case (see [15] and the discussion in section 2), it may be shown that the Poisson bracket of the form

$$\left\{\widehat{\varphi}, \widehat{\psi}\right\} = \left\langle\!\!\left\langle \begin{pmatrix} r & -\hat{h}r_+ \\ \hat{h}^{-1}r_- & -r \end{pmatrix} \begin{pmatrix} \nabla\widehat{\varphi} \\ \nabla'\widehat{\varphi} \end{pmatrix}, \begin{pmatrix} \nabla\widehat{\psi} \\ \nabla'\widehat{\psi} \end{pmatrix} \right\rangle\!\!\right\rangle \tag{5.1}$$

is invariant with respect to the gauge action, and that, moreover, the gauge action of $L\mathfrak{N}_+^\lambda$ is admissible. We put $Z_{\widehat{\varphi}} = {}^{\hat{h}^{-1}}\nabla\widehat{\varphi} - \nabla'\widehat{\varphi}$, $\bar{Z}_{\widehat{\varphi}} = {}^{\hat{h}^{-1}}\nabla\widehat{\varphi} + \nabla'\widehat{\varphi}$ and rewrite (5.1) as follows:

$$\left\{\widehat{\varphi}, \widehat{\psi}\right\} = \left\langle Z_{\widehat{\varphi}}, \tfrac{1}{2}\bar{Z}_{\widehat{\psi}} - rZ_{\widehat{\psi}}\right\rangle. \tag{5.2}$$

Let us define $\hat{s} \in \text{End}L\mathfrak{h}^\lambda$ by

$$\hat{s}\,\text{diag}(f_0, f_1, \dots) = \text{diag}(f_1, f_2, \dots). \tag{5.3}$$

Proposition 5.1. *Condition 2) above is equivalent to the following relation for r_0:*

$$r_0\left(1 - \hat{h}\,\hat{s}\right)f = \tfrac{1}{2}\left(1 + \hat{h}\,\hat{s}\right)f + \alpha(f), \quad \forall f \in \Lambda\left(L\mathcal{S}_1^\lambda\right), \tag{5.4}$$

where $\alpha(\cdot) : \mathfrak{h}^\lambda \to \mathbb{C}\cdot 1 \subset \mathfrak{h}^\lambda$ is a linear operator.

Proof. Let $\widehat{\varphi}, \widehat{\psi}$ be $L\mathfrak{N}_+^\lambda$ -invariant functions, $\widehat{\psi}_{|\mathbb{Y}_q^\lambda} = \text{const}$; it is easy to see that $Z_{\widehat{\varphi}}, Z_{\widehat{\psi}} \in L\mathfrak{b}_+^\lambda$ and for any $\mathcal{L} \in \mathbb{Y}_q^\lambda$ the gradient $d\widehat{\psi}(\mathcal{L}) \in L\mathfrak{n}_+^\lambda$, which implies that $\nabla\widehat{\varphi}, \nabla'\widehat{\varphi}, \bar{Z}_{\widehat{\psi}} \in L\mathfrak{b}_+^\lambda$ on $\mathbb{Y}_q^\lambda$. Taking into account that $L\mathfrak{n}_+^\lambda$ is isotropic and $\left(L\mathfrak{h}^\lambda\right)^* \simeq L\mathfrak{h}^\lambda$, we obtain

$$\left\{\widehat{\varphi}, \widehat{\psi}\right\}(\mathcal{L}) = \left\langle Z_{\widehat{\varphi}}^0, \tfrac{1}{2}\bar{Z}_{\widehat{\psi}}^0 - r_0 Z_{\widehat{\psi}}^0\right\rangle, \quad \mathcal{L} \in \mathbb{Y}_q^\lambda,$$

where $Z_{\widehat{\varphi}}^0 = \mathcal{P}_0 Z_{\widehat{\varphi}}$. But $\left\{\widehat{\varphi}, \widehat{\psi}\right\} = 0$ on $\mathbb{Y}_q^\lambda$, hence $\tfrac{1}{2}\bar{Z}_{\widehat{\psi}}^0 - r_0 Z_{\widehat{\psi}}^0$ is orthogonal to all $Z_{\widehat{\varphi}}^0$.

Lemma 5.2. *For any $f \in L\mathfrak{h}^\lambda$ with $\int \frac{dz}{z} Tr\, f = 0$ there exists an $L\mathfrak{N}_+^\lambda$-invariant function $\widehat{\varphi}$ and $\mathcal{L} \in \mathbb{Y}_q^\lambda$ such that $Z_{\widehat{\varphi}}^0 = f$.*

This lemma implies that

$$r_0 Z_{\widehat{\psi}}^0 = \tfrac{1}{2} \bar{Z}_{\widehat{\psi}}^0 + \tilde{\alpha}\left(d\widehat{\psi}\right), \tag{5.5}$$

where $\tilde{\alpha}\left(\cdot\right) : L\mathfrak{gl}_q^\lambda \to \mathbb{C} \cdot 1 \subset L\mathfrak{h}^\lambda$ is some linear operator.

Then, it is easy to see that only $L\mathcal{S}_1^\lambda$-component of $d\widehat{\psi}$ gives contribution in $\bar{Z}_{\widehat{\psi}}^0$, $Z_{\widehat{\psi}}^0$:

$$\begin{aligned}
Z_{\widehat{\psi}}^0 &= \left(1 - \hat{h}\,\hat{s}\right)\Lambda d\widehat{\psi}^{(1)}, \\
\bar{Z}_{\widehat{\psi}}^0 &= \left(1 + \hat{h}\,\hat{s}\right)\Lambda d\widehat{\psi}^{(1)}.
\end{aligned} \tag{5.6}$$

Also it is evident that

$$\tilde{\alpha}\left(d\widehat{\psi}\right) = \alpha\left(\Lambda d\widehat{\psi}^{(1)}\right), \tag{5.7}$$

where $\alpha\left(\cdot\right)$ is a linear operator in $L\mathfrak{h}^\lambda$ with $\mathrm{Im}\,\alpha \subset \mathbb{C}\cdot 1 \subset L\mathfrak{h}^\lambda$. Substituting (5.6) and (5.7) into (5.5) we obtain (5.4).∎

Remark 5.1. *Proposition 5.1 shows the following important difference between the $\mathfrak{gl}_q^\lambda$-case and the finite-dimensional cases considered in [8, 17, 14].*

In the $\mathfrak{gl}_n$-case the diagonal component $\hat{r}^0$ of the r-matrix is given (up to a skew-symmetric operator in U_n) by the Cayley transformation of $\hat{h}\tau_n$, where τ_n acts in the subspace of diagonal matrices by cyclic permutation of matrix elements (see (2.10)). Obviously, it is impossible to define an analog of τ_n in the $\mathfrak{gl}_q^\lambda$-case. It is replaced now by the shift operator $\hat{s}$, whose properties are quite different. This causes some difficulties. As we shall see below, the operator $A \equiv 1 - \hat{h}\,\hat{s}$ is not invertible, its kernel is isomorphic to $\mathbb{C}\left((z^{-1})\right)$. However, $\mathrm{Im}\,A = L\mathfrak{h}^\lambda$ (this is possible only in the infinite-dimensional case), so we can define a regularized operator A^{-1} and find a r-matrix satisfying (5.4).

We shall now study the properties of A. We define the following subspaces in $L\mathfrak{h}^\lambda$:

$$\begin{aligned}
V^\lambda &= \left\{f \in L\mathfrak{h}^\lambda : f = \mathrm{diag}\left(0, *, \dots\right)\right\}; \\
H_1^\lambda &= \left\{f \in L\mathfrak{h}^\lambda : f\left(\lambda\right) = 0\right\}; \\
V_1^\lambda &= V^\lambda \cap H_1^\lambda; \\
\mathrm{Im}V_1^\lambda &= \left(1 - \hat{h}\,\hat{s}\right)V_1^\lambda; \\
U &= \mathrm{Ker}\left(1 - \hat{h}\,\hat{s}\right).
\end{aligned} \tag{5.8}$$

Below we assume that $\lambda \neq 0$.

Proposition 5.3.

1. $U = \left\{\mathrm{diag}\left(F_0\left(z\right), F_0\left(q^{-1}z\right), \dots\right), \quad F_0\left(z\right) \in \mathbb{C}\left((z^{-1})\right)\right\}.$

2. *The restriction of the operator A to V^λ*

$$A_{|V^\lambda} : V^\lambda \to L\mathfrak{h}^\lambda$$

is a bijection. We shall denote $A^{-1} \equiv \left[A_{|V^\lambda}\right]^{-1}$; it is given by

$$\left(A^{-1}F\right)_n (z) = -\sum_{i=0}^{n-1} F_i \left(q^{i-n}z\right), \quad \forall F = \mathrm{diag}\left(F_0\left(z\right), F_1\left(z\right), \dots\right), \quad n \in \mathbb{N}. \tag{5.9}$$

3. *$V^\lambda = V_1^\lambda \dotplus A^{-1}U$.*
4. *With respect to the invariant inner product $L\mathfrak{h}^\lambda$ is the orthogonal sum*

$$L\mathfrak{h}^\lambda = \mathrm{Im}V_1^\lambda \oplus U; \tag{5.10}$$

the projection operator on U is given by

$$\left(\mathcal{P}_U F\right)_0 (z) = -\frac{1}{\lambda}\left(A^{-1}F\right)\left(\lambda, q^\lambda z\right), \tag{5.11}$$

(obviously, an element of U is uniquely defined by its 0-th component).

Proof. Assertions 1, 2 are evident, they result directly from the definition of A. To prove 3 suppose that there exists $f \in V_1^\lambda \cap A^{-1}U$. Put $f = A^{-1}F$; obviously, $F \in U$. Assertion 1 and (5.9) imply that $\left(A^{-1}F\right)_n (z) = -nF_0\left(q^{-n}z\right)$ and hence

$$\left(A^{-1}F\right)\left(\lambda, z\right) = -\lambda F_0 \left(q^{-\lambda}z\right). \tag{5.12}$$

But $\left(A^{-1}F\right) \equiv f\left(\lambda, z\right) = 0$, since $f \in V_1^\lambda$, hence $F_0 = 0$, and $f = 0$. So, the sum $V_1^\lambda \dotplus A^{-1}U$ is direct. Let us prove that $V^\lambda \subset V_1^\lambda \dotplus A^{-1}U$ (the inclusion $\supset$ is evident). For $f \in V^\lambda$ we choose $F \in U$ defined by its component $F_0 (z) = -\frac{1}{\lambda}f\left(\lambda, q^\lambda z\right)$. The (5.12) implies that $\left(f - A^{-1}F\right)\left(\lambda, z\right) = 0$, i.e. $f - A^{-1}F \in V_1^\lambda$ as desired.

Assertions 2 and 3 imply that the sum (5.10) is direct, formula (5.11) follows directly from (5.12). It remains to verify that $\mathrm{Im}V_1^\lambda \perp U$. Recall the definition of Tr on $L\mathfrak{h}^\lambda$: $Tr f(z) = \mathbb{D}_f\left(\lambda, z\right)$, where $\mathbb{D}_f\left(\lambda, z\right)$ is an $\mathcal{A}_0$-function uniquely defined by $\mathbb{D}_f (n, z) = \sum_{i=0}^{n-1} f_i (z)$. Let $f \in V_1^\lambda$, $g \in U$. We have

$$\begin{aligned}\langle Af, g\rangle = \left\langle\left(1 - \hat{h}\,\hat{s}\right)f, g\right\rangle &= \int \frac{dz}{z}\left[Tr\, fg - Tr\left(\hat{s}f \cdot {}^{h^{-1}}g\right)\right] \\ &= \int \frac{dz}{z}\left[\mathbb{D}_{fg} - \mathbb{D}_{\hat{s}f \cdot {}^{h^{-1}}g}\right]\left(\lambda, z\right).\end{aligned}$$

But

$$\left[\mathbb{D}_{fg} - \mathbb{D}_{\hat{s}f.\,h^{-1}g} \right] (n, z) = \sum_{i=0}^{n-1} \left[f_i(z)\, g_i(z) - f_{i+1}(z)\, g_i\left(q^{-1}z\right) \right]$$

$$= \sum_{i=0}^{n-1} f_i(z) \left[g_i(z) - g_{i-1}\left(q^{-1}z\right) \right] \quad \text{(because } f_0 = 0\text{)}$$

$$= \sum_{i=0}^{n-1} f_i(z) \left[g_0\left(q^{-i}z\right) - g_0\left(q^{-(i-1)}q^{-1}z\right) \right] = 0$$

as desired. ∎

Remark 5.2. *As above (see remark 2.3), it can be proved that the terms containing α do not affect the corresponding Poisson bracket. Below we put $\alpha = 0$.*

Taking this into account, we have the following

Theorem 5.4. *A Poisson bracket on Lgl_q^λ of the form (5.1) admits the generalized DS-reduction if and only if the corresponding r-matrix r_0 is chosen in the form*

$$r_0 = -\tfrac{1}{2} + A^{-1} + \left(\bar{B} + \tfrac{\lambda}{2} \right) \mathcal{P}_U, \tag{5.13}$$

where $\bar{B}$ is a skew-symmetric linear operator in U.

Remark 5.3. *The subspace U can be naturally identified with $\mathbb{C}\left(\left(z^{-1}\right)\right)$ (by taking the 0-th component $F_0(z)$ of $F \in U$). Hence we may consider the operator $\bar{B}$ as a skew-symmetric linear operator in $\mathbb{C}\left(\left(z^{-1}\right)\right)$.*

Proof of theorem 5.4. It is easy to see that $\Lambda\left(L\mathcal{S}_1^\lambda\right) = V_1^\lambda$. Then proposition 5.3 shows that any operator r_0 satisfying (5.4) can be written in the form

$$r_0 = \frac{1}{2}\left(1 + \hat{h}\,\hat{s}\right) A^{-1} + \tilde{B}\mathcal{P}_U, \tag{5.14}$$

where $\tilde{B}$ is a linear operator, $\tilde{B} : U \to L\mathfrak{h}^\lambda$. The theorem follows directly from

Proposition 5.5. *The skew-symmetry of r_0 is equivalent to the following conditions:*
1) $\mathrm{Im}\tilde{B} \subset U$,
2) $\tilde{B} + \tilde{B}^* = \lambda$.

Proof. The skew-symmetry of r_0 means that for any $f, g \in L\mathfrak{h}^\lambda$

$$\langle r_0 f, g \rangle + \langle f, r_0 g \rangle = 0. \tag{5.15}$$

By proposition 5.3, any $f \in L\mathfrak{h}^\lambda$ has a unique decomposition of the form

$$f = A\hat{f} + \bar{f}, \quad \hat{f} \in V_1^\lambda, \quad \bar{f} \in U. \tag{5.16}$$

Substituting this in (5.15) and taking into account that $\frac{1}{2}\left(1+\hat{h}\,\hat{s}\right)A^{-1} \equiv -\frac{1}{2} + A^{-1}$, we obtain

$$\langle r_0 f, g\rangle + \langle f, r_0 g\rangle = I_1\left(\hat{f}, \hat{g}\right) + I_2\left(\hat{f}, \bar{g}\right) + I_2\left(\hat{g}, \bar{f}\right) + I_3\left(\bar{f}, \bar{g}\right),$$

where

$$I_1\left(\hat{f}, \hat{g}\right) = -\left\langle A\hat{f}, A\hat{g}\right\rangle + \left\langle \hat{f}, A\hat{g}\right\rangle + \left\langle A\hat{f}, \hat{g}\right\rangle, \tag{5.17}$$

$$I_2\left(\hat{f}, \bar{g}\right) = \left\langle \hat{f}, \bar{g}\right\rangle + \left\langle A\hat{f}, A^{-1}\bar{g}\right\rangle + \left\langle A\hat{f}, \tilde{B}\bar{g}\right\rangle, \tag{5.18}$$

$$I_3\left(\bar{f}, \bar{g}\right) = -\left\langle \bar{f}, \bar{g}\right\rangle + \left\langle A^{-1}\bar{f}, \bar{g}\right\rangle + \left\langle \bar{f}, A^{-1}\bar{g}\right\rangle + \left\langle \tilde{B}\bar{f}, \bar{g}\right\rangle + \left\langle \bar{f}, \tilde{B}\bar{g}\right\rangle \tag{5.19}$$

Recalling that $A \equiv 1 - \hat{h}\,\hat{s}$ and $\hat{h}^* = \hat{h}^{-1}$ we find

$$I_1\left(\hat{f}, \hat{g}\right) = \left\langle \hat{f}, \hat{g}\right\rangle - \left\langle \hat{s}\hat{f}, \hat{s}\hat{g}\right\rangle.$$

Lemma 5.6. *For any* $\hat{f}, \hat{g} \in V^\lambda$

$$\left\langle \hat{f}, \hat{g}\right\rangle - \left\langle \hat{s}\hat{f}, \hat{s}\hat{g}\right\rangle = -\int \frac{dz}{z}\hat{f}\left(\lambda, z\right)\hat{g}\left(\lambda, z\right).$$

In our case $\hat{f}, \hat{g} \in V_1^\lambda$ i.e. $\hat{f}\left(\lambda, z\right) = \hat{g}\left(\lambda, z\right) = 0$, hence $I_1\left(\hat{f}, \hat{g}\right) = 0$, as desired.

Now, the skew-symmetry of r_0 implies that both I_2 and I_3 are equal to zero.

The first two terms in (5.18) vanish; indeed, taking into account that $\left\langle A\hat{f}, \bar{g}\right\rangle = 0$, we have

$$\left\langle \hat{f}, \bar{g}\right\rangle + \left\langle A\hat{f}, A^{-1}\bar{g}\right\rangle = \left\langle \hat{f}, \bar{g}\right\rangle + \left\langle A\hat{f}, A^{-1}\bar{g}\right\rangle - \left\langle A\hat{f}, \bar{g}\right\rangle = I_1\left(\hat{f}, A^{-1}\bar{g}\right) = 0.$$

Hence, $I_2 = 0$ implies $\left\langle A\hat{f}, \tilde{B}\bar{g}\right\rangle = 0$, i.e. $\mathrm{Im}\tilde{B} \perp \mathrm{Im}V_1^\lambda$. Then by proposition 5.3 $\mathrm{Im}\tilde{B} \subset U$, as desired.

Recall now that by (5.12) $\left(A^{-1}\bar{f}\right)\left(\lambda, z\right) = -\lambda\bar{f}_0\left(q^{-\lambda}z\right)$, hence

$$\begin{aligned}
0 &= I_3\left(\bar{f}, \bar{g}\right) = -\left\langle \bar{f}, \bar{g}\right\rangle + \left\langle A^{-1}\bar{f}, \bar{g}\right\rangle + \left\langle \bar{f}, A^{-1}\bar{g}\right\rangle + \left\langle \tilde{B}\bar{f}, \bar{g}\right\rangle + \left\langle \bar{f}, \tilde{B}\bar{g}\right\rangle \\
&= I_1\left(A^{-1}\bar{f}, A^{-1}\bar{g}\right) + \left\langle \tilde{B}\bar{f}, \bar{g}\right\rangle + \left\langle \bar{f}, \tilde{B}\bar{g}\right\rangle \\
&= -\int \frac{dz}{z}\left(A^{-1}\bar{f}\right)\left(\lambda, z\right)\left(A^{-1}\bar{g}\right)\left(\lambda, z\right) + \left\langle \tilde{B}\bar{f}, \bar{g}\right\rangle + \left\langle \bar{f}, \tilde{B}\bar{g}\right\rangle \\
&= -\lambda^2\int \frac{dz}{z}\bar{f}_0\left(q^{-\lambda}z\right)\bar{g}_0\left(q^{-\lambda}z\right) + \left\langle \tilde{B}\bar{f}, \bar{g}\right\rangle + \left\langle \bar{f}, \tilde{B}\bar{g}\right\rangle \\
&= -\lambda\left\langle \bar{f}, \bar{g}\right\rangle + \left\langle \tilde{B}\bar{f}, \bar{g}\right\rangle + \left\langle \bar{f}, \tilde{B}\bar{g}\right\rangle,
\end{aligned}$$

i.e. $\tilde{B} + \tilde{B}^* = \lambda$. $\blacksquare$

6. Explicit formula for the quotient bracket and uniqueness theorem.

In this section we give an explicit formula for the brackets obtained via DS-redaction on the quotient $\mathbb{Y}_q^\lambda/LN_+^\lambda$ which may be identified with the set of q-pseudodifference operators of complex degree λ. We shall see that only one of them satisfies involutivity condition (2.40). This bracket coincides with the one constructed in [14].

In this section we assume that λ is generic, i.e. $\lambda \notin \frac{2\pi i}{\ln q}\mathbb{Q}$.

As shown in section 2, the quotient $\mathbb{Y}_q^\lambda/LN_+^\lambda$ may be identified with the set $\mathbb{Y}_0$ of companion matrices, i.e. the matrices of the form

$$\widetilde{\mathcal{L}} = \begin{pmatrix} -u_1(z) & -u_2(z) & -u_3(z) & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \vdots & \vdots & \ddots & \dots \end{pmatrix}, \quad u_i(z) \in \mathbb{C}((z^{-1})). \tag{6.1}$$

As an affine space $\mathbb{Y}_0$ is isomorphic to the set $\widehat{\mathbb{G}}_\lambda$ of q-pseudodifference operators of a complex degree λ:

$$\widehat{\mathbb{G}}_\lambda = \left\{ L = D^\lambda + u_1(z) D^{\lambda-1} + u_2(z) D^{\lambda-2} + \cdots \right\}. \tag{6.2}$$

Note also that all $\widehat{\mathbb{G}}_\lambda$ are isomorphic (as affine spaces) to each other and to $\prod_{i \geq 1} \mathbb{C}((z^{-1}))$; so

$$\mathbb{Y}_q^\lambda/LN_+^\lambda \simeq \mathbb{Y}_0 \simeq \widehat{\mathbb{G}}_\lambda \simeq \prod_{i \geq 1} \mathbb{C}((z^{-1})).$$

We fix the following models of the tangent and cotangent spaces of $\widehat{\mathbb{G}}_\lambda$:

$$\begin{aligned} T_L\widehat{\mathbb{G}}_\lambda &= \left\{ X = \bar{X} D^\lambda : \bar{X} \in J_- \subset \Psi D_q \right\}, \\ T_L^*\widehat{\mathbb{G}}_\lambda &= \left\{ f = D^{-\lambda}\bar{f} : \bar{f} \in J_+ \subset \Psi D_q \right\}. \end{aligned} \tag{6.3}$$

The canonical pairing between $T_L\widehat{\mathbb{G}}_\lambda$ and $T_L^*\widehat{\mathbb{G}}_\lambda$ is given by

$$\langle X, f \rangle = Tr_{\Psi D_q} X \cdot f. \tag{6.4}$$

The space $Fun\left(\widehat{\mathbb{G}}_\lambda\right)$ of the smooth functional on $\widehat{\mathbb{G}}_\lambda$ is generated by the Laurent coefficients u_i^m of the functions $u_i(z)$. The left and right gradients of a functional $\varphi \in Fun\left(\widehat{\mathbb{G}}_\lambda\right)$ are defined by the usual formulas:

$$\nabla\varphi(L) = Ld\varphi, \quad \nabla'\varphi = d\varphi L, \quad d\varphi(L) \in T_L^*\widehat{\mathbb{G}}_\lambda, \quad L \in \widehat{\mathbb{G}}_\lambda.$$

It is easy to see that the left and right gradients contain only integer powers of D and therefore may be considered as elements of ΨD_q.

We consider the class of Poisson brackets on $\widehat{\mathbb{G}}_\lambda$ of the form

$$\{\varphi, \psi\} = \left\langle\!\!\left\langle \left(\begin{array}{cc} R + aP_0 & bP_0 \\ cP_0 & R + dP_0 \end{array} \right) D\varphi, D\psi \right\rangle\!\!\right\rangle, \qquad (6.5)$$

where

$$D\varphi \equiv \left(\begin{array}{c} \nabla\varphi \\ \nabla'\varphi \end{array} \right)$$

and similarly for $D\psi$, $R = \frac{1}{2}(P_+ - P_-)$ and a, b, c, d are linear operators in $J_0 \simeq \mathbb{C}((z^{-1})) \subset \Psi D_q$ satisfying the skew-symmetry conditions

$$a = -a^*, \quad d = -d^*, \quad c^* = b.$$

Remark 6.1. *Note that for a functional $\varphi \in Fun\left(\widehat{\mathbb{G}}_\lambda\right)$ its linear gradient $d\varphi$ is defined up to an arbitrary element of $D^{-\lambda}J_{(-)}$; in (6.3) we have put $D^{-\lambda}d\varphi \in D^{-\lambda}J_+$, but it is a manually imposed restriction. The bracket (6.5) is said to be well-defined if its value does not depend on the $D^{-\lambda}J_{(-)}$-components of $d\varphi, d\psi$. It is easy to see that the bracket (6.5) is well-defined if and only if*

$$a + \tfrac{1}{2} + bD^{-\lambda} = c + \left(\tfrac{1}{2} + d\right)D^{-\lambda} = \alpha\,Tr\,\cdot, \quad \alpha \in \mathbb{C}. \qquad (6.6)$$

Let $\mathcal{P}_{00}$ be the projection operator on the one-dimensional subspace in U generated by the unity matrix, and $\mathcal{P}_0' = \mathcal{P}_0 - \mathcal{P}_{00}$.

Theorem 6.1. *Let*

$$r_0^{\lambda,\Delta} = -\tfrac{1}{2} + A^{-1} + \left(\bar{B}^{\lambda,\Delta} + \tfrac{\lambda}{2}\right)\mathcal{P}_U, \quad \bar{B}^{\lambda,\Delta} = \lambda\left(\tfrac{1}{2}\tfrac{1+\hat{h}^\lambda}{1-\hat{h}^\lambda}\mathcal{P}_0' + \Delta\right),$$
$$\qquad (6.7)$$

where Δ is a skew-symmetric operator in U, commuting with $\hat{h}$. Let $r^{\lambda,\Delta} = \frac{1}{2}(\mathcal{P}_+ - \mathcal{P}_-) + r_0^{\lambda,\Delta}\mathcal{P}_0$.

The Poisson bracket $\{\cdot, \cdot\}_\Delta^\lambda$ on $L\mathfrak{gl}_q^\lambda$ defined by

$$\left\{\widehat{\varphi}, \widehat{\psi}\right\}_\Delta^\lambda = \left\langle\!\!\left\langle \left(\begin{array}{cc} r^{\lambda,\Delta} & -\hat{h}r_+^{\lambda,\Delta} \\ \hat{h}^{-1}r_-^{\lambda,\Delta} & -r^{\lambda,\Delta} \end{array} \right)\left(\begin{array}{c} \nabla\widehat{\varphi} \\ \nabla'\widehat{\varphi} \end{array} \right), \left(\begin{array}{c} \nabla\widehat{\psi} \\ \nabla'\widehat{\psi} \end{array} \right) \right\rangle\!\!\right\rangle,$$
$$\qquad (6.8)$$

gives rise via DS-reduction to the following bracket on $\widehat{\mathbb{G}}_\lambda$:

$$\{\varphi, \psi\}_\lambda^\Delta = \left\langle\!\!\left\langle \left(\begin{array}{cc} R + \left(\tfrac{1}{2}\tfrac{1+\hat{h}^\lambda}{1-\hat{h}^\lambda} + \Delta\right)P_0' & -\left(\tfrac{1}{1-\hat{h}^\lambda} + \Delta\right)\hat{h}^\lambda P_0' \\ \left(\tfrac{\hat{h}^\lambda}{1-\hat{h}^\lambda} + \Delta\right)\hat{h}^{-\lambda}P_0' & R - \left(\tfrac{1}{2}\tfrac{1+\hat{h}^\lambda}{1-\hat{h}^\lambda} + \Delta\right)P_0' \end{array} \right) D\varphi, D\psi \right\rangle\!\!\right\rangle;$$
$$\qquad (6.9)$$

(here we have identified $EndU$ and $End\mathbb{C}((z^{-1}))$.)

Like to the finite-dimensional case, the corresponding r-matrix may be uniquely fixed by imposing, in addition, the involutivity condition:

Theorem 6.2. *There exists a unique bracket of the form (6.8) on $L\mathfrak{gl}_q^\lambda$ which admits DS-reduction and gives rise to a Poisson bracket on $\widehat{\mathbb{G}}_\lambda$ satisfying the involutivity condition*

$$a + b = c + d. \tag{6.10}$$

This bracket coincides with $\{\cdot, \cdot\}_0^\lambda$.

The proof is similar to the one of theorem 2.11 and will be omitted.

The rest of this section is devoted to the proof of theorem 6.1. The general idea of the proof is similar to the one of [11]: both brackets (6.8) and (6.9) considered as functions of λ are quotients of two $\mathcal{A}_0$-functions; therefore, it is sufficient to prove the theorem for all sufficiently large integer $\lambda = N$. But in this case the DS-reduction on $L\mathfrak{gl}_q^\lambda$ amounts to the q-deformed DS-reduction on $L\mathfrak{gl}_N$ considered in section 2, for which theorem 2.10 gives formula (6.9).

Let us define filtrations on the spaces $Fun\left(\mathbb{Y}_q^\lambda\right)$ and $Fun\left(\widehat{\mathbb{G}}_\lambda\right)$. Recall that we have denoted by $LS_i \subset L\mathfrak{gl}_q^\lambda$ the subspace of matrices which have only i-th non-zero diagonal. Let $\mathbb{V}_n = \bigoplus_{i=0}^{n-1} LS_i$, let $\Gamma_n : \mathbb{Y}_q^\lambda \to \mathbb{V}_n$ be the natural projection and $\Omega_n = \Gamma_n^*\left(Fun\left(\mathbb{V}_n\right)\right).$ Obviously,

$$\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega_n \subset \ldots \subset Fun\left(\mathbb{Y}_q^\lambda\right), \quad \bigcup_{i \geq 1} \Omega_i = Fun\left(\mathbb{Y}_q^\lambda\right). \tag{6.11}$$

Lemma 6.3. *For any Poisson bracket of the form (6.8)*

$$\{\Omega_i, \Omega_j\}_\Delta^\lambda \subset \Omega_{i+j}.$$

As it was noted $\widehat{\mathbb{G}}_\lambda \simeq \prod_{i \geq 1} \mathbb{C}((z^{-1}))$ as affine spaces. Let $\mathbb{M}_n = \prod_{i=1}^{n} \mathbb{C}((z^{-1}))$, let $\gamma_n : \prod_{i \geq 1} \mathbb{C}((z^{-1})) \to \prod_{i=1}^{n} \mathbb{C}((z^{-1}))$ be the natural projection and $W_n = \gamma_n^*\left(Fun\left(\mathbb{M}_n\right)\right).$ We have

$$W_1 \subset W_2 \subset \ldots \subset W_n \subset \ldots \subset Fun\left(\widehat{\mathbb{G}}_\lambda\right), \quad \bigcup_{i \geq 1} W_i = Fun\left(\widehat{\mathbb{G}}_\lambda\right). \tag{6.12}$$

Lemma 6.4. *For any Poisson bracket of the form (6.9)*

$$\{W_i, W_j\}_\lambda^\Delta \subset W_{i+j}.$$

The proof of the cross-section theorem 4.1 implies that these filtrations are consistent with the projection $\pi : \mathbb{Y}_q^\lambda \to \widehat{\mathbb{G}}_\lambda$, i.e. $\pi^*\left(W_i\right) \subset \Omega_i$.

Let us choose some functions $\varphi \in W_i$, $\psi \in W_j$. Let $\widehat{\varphi} = \pi^*\varphi$, $\widehat{\psi} = \pi^*\psi$; they are defined on $\mathbb{Y}_q^\lambda$ and can be continued to LN_+^λ-invariant functions on the whole

$L\mathfrak{gl}_q^\lambda$ which will be denoted by the same letters. We need to prove that for any $L \in \widehat{\mathbb{G}}_\lambda$ and some (and then for any) $\mathcal{L} \in \pi^{-1}(L)$

$$\left\{\widehat{\varphi}, \widehat{\psi}\right\}_\Delta^\lambda (\mathcal{L}) = \{\varphi, \psi\}_\lambda^\Delta (L). \tag{6.13}$$

We choose $\mathcal{L} \in \pi^{-1}(L)$ in the companion form (6.1). Since $\left\{\widehat{\varphi}, \widehat{\psi}\right\}_\Delta^\lambda \in \Omega_{i+j}$ and $\{\varphi, \psi\}_\lambda^\Delta \in W_{i+j}$ we may assume that $L \in \mathbb{M}_{i+j}$.

Lemma 6.5. *If (6.13) holds for all sufficiently large integer λ, then it holds for all generic $\lambda \in \mathbb{C}$.*

Proof. Since all $\widehat{\mathbb{G}}_\lambda$ are isomorphic to each other and to $\prod_{i \geq 1} \mathbb{C}((z^{-1}))$, we may consider $\{\varphi, \psi\}_\lambda^\Delta (L)$ as a function of variable λ. Functionals φ, ψ depend only on a finite number of Laurent coefficients u_i^n, hence from (6.9) it follows that $\{\varphi, \psi\}_\lambda^\Delta (L)$ is the quotient of two $\mathcal{A}_0$-functions.

The bracket $\left\{\widehat{\varphi}, \widehat{\psi}\right\}_\Delta^\lambda (\mathcal{L})$ may also be considered as a function of λ because the set $\mathbb{Y}_0$ of companion matrices is naturally embedded into $L\mathfrak{gl}_q^\lambda$ for all $\lambda \in \mathbb{C}$. It is easy to see that this bracket is also the quotient of two $\mathcal{A}_0$-functions. Then the lemma follows directly from the interpolation property for $\mathcal{A}_0$-functions (see proposition 3.1). ∎

Let us fix some $\lambda = m > i + j$. The subspace $\mathbb{M}_m \subset \widehat{\mathbb{G}}_m$ is Poisson (see [14]). It is naturally identified with the affine space of normalized m-th order q-difference operators defined in section 2 which was denoted there by the same letter (see (2.2)). The functions φ, ψ may be considered as functions on $\mathbb{M}_m$. As discussed in section 2, the bracket $\{\cdot, \cdot\}_m^\Delta$ on $\mathbb{M}_m$ can be obtained via the ordinary q-deformed DS-reduction from $L\mathfrak{gl}_m$. Let $\rho : \mathbb{Y}_m \to \mathbb{Y}_m/LN_+(m) \simeq \mathbb{M}_m$ be the corresponding projection, let $\bar{\varphi}, \bar{\psi}$ be $LN_+(m)$-invariant functions on $L\mathfrak{gl}_m$ such that $\bar{\varphi}_{|\mathbb{Y}_m} = \rho^*\varphi$, $\bar{\psi}_{|\mathbb{Y}_m} = \rho^*\psi$. Theorem 2.10 says that

$$\left\{\bar{\varphi}, \bar{\psi}\right\}_{\Delta,m} (\mathbb{L}) = \{\varphi, \psi\}_m^\Delta (L),$$

where $\mathbb{L} \in L\mathfrak{gl}_m$ is a companion matrix corresponding to L. Hence, we need to prove only that

$$\left\{\widehat{\varphi}, \widehat{\psi}\right\}_\Delta^m (\mathcal{L}) = \left\{\bar{\varphi}, \bar{\psi}\right\}_{\Delta,m} (\mathbb{L}). \tag{6.14}$$

Recall that we have defined an embedding of $\mathfrak{gl}_m$ into $\mathfrak{gl}_q^\lambda$ (see (3.6)); we may naturally define a similar embedding for the corresponding loop algebras. As above, for an element $A \in L\mathfrak{gl}_q^\lambda$ we denote by $A_{|m}$ its $\mathfrak{gl}_m$-block. The similar notation will be used also for subspaces: for $K \subset L\mathfrak{gl}_q^\lambda$ we put

$$K_{|m} = \left\{A_{|m} : \forall A \in K\right\}.$$

The linear gradient $d\widehat{\varphi}$ is defined up to an arbitrary element of $L\mathfrak{n}_+^m$; however, this freedom does not affect the value of the bracket. We shall assume that $(d\widehat{\varphi})_+ = 0$, so

$$d\widehat{\varphi} \in \bigoplus_{k=-i}^{0} L\mathcal{S}_k \subset L\mathfrak{b}_-^m, \tag{6.15}$$

because $\widehat{\varphi} \in \Omega_i$. For $d\bar{\varphi}$ we shall also use the assumption $(d\bar{\varphi})_+ = 0$.

Lemma 6.6.

$$\left\{\widehat{\varphi}, \widehat{\psi}\right\}_\Delta^m (\mathcal{L}) = \left\langle Z_{\bar{\varphi}}(\mathbb{L}), \tfrac{1}{2}\bar{Z}_{\bar{\psi}}(\mathbb{L}) - \left(r^{m,\Delta}\right)_{|m} Z_{\bar{\psi}}(\mathbb{L}) \right\rangle. \tag{6.16}$$

Proof. It is easy to see that

$$d\widehat{\varphi}_{|m}(\mathcal{L}) = d\bar{\varphi}(\mathbb{L}); \tag{6.17}$$

(this follows from the fact that both in $\mathfrak{gl}_m$- and in $\mathfrak{gl}_q^\lambda$-cases the gauge action of the upper triangular group is free and that the restriction of the $\mathfrak{gl}_q^\lambda$-trace to $\mathfrak{gl}_m$ coincides with the ordinary matrix trace).

Using (6.15), (6.17) and the evident relation $\mathcal{L}_{|m} = \mathbb{L}$ we find by direct computation that left and right gradients $\nabla\widehat{\varphi}, \nabla'\widehat{\varphi}$ have the form

and that $\nabla\widehat{\varphi}_{|m} = \nabla\bar{\varphi}$, $\nabla'\widehat{\varphi}_{|m} = \nabla'\bar{\varphi}$.

Obviously, $Z_{\widehat{\varphi}}$, $\bar{Z}_{\widehat{\varphi}}$ have a similar form; but $Z_{\widehat{\varphi}}$ is upper-triangular, hence

$$Z_{\widehat{\varphi}} = \begin{pmatrix} Z_{\bar{\varphi}} & 0 \\ 0 & 0 \end{pmatrix}. \tag{6.18}$$

Substituting this into

$$\left\{\widehat{\varphi}, \widehat{\psi}\right\}_\Delta^m (\mathcal{L}) = \left\langle Z_{\widehat{\varphi}}, \tfrac{1}{2}\bar{Z}_{\widehat{\psi}} - r^{m,\Delta} Z_{\widehat{\psi}} \right\rangle$$

we obtain (6.16). ∎

On the other hand, by definition,

$$\left\{\bar{\varphi}, \bar{\psi}\right\}_{\Delta,m}(\mathbb{L}) = \left\langle Z_{\bar{\varphi}}, \tfrac{1}{2}\bar{Z}_{\bar{\psi}} - \hat{r}_{\Delta,m} Z_{\bar{\psi}} \right\rangle, \tag{6.19}$$

where

$$\hat{r}_{\Delta,m} = \frac{1}{2}\left(\mathcal{P}_+ - \mathcal{P}_-\right) + \hat{r}_{\Delta,m}^0, \quad \hat{r}_{\Delta,m}^0 = \frac{1}{2}\frac{1+\hat{h}\tau_m}{1-\hat{h}\tau_m}\mathcal{P}_0' + m\Delta\mathcal{P}_{U_m}. \tag{6.20}$$

(Recall that $U_m = \left\{\mathrm{diag}\left(f_0\left(z\right), f_0\left(q^{-1}z\right), \ldots, f_0\left(q^{-(m-1)}z\right)\right)\right\} = U_{|m}$). The following lemma finishes our proof:

Lemma 6.7.

$$\left(r_0^{m,\Delta}\right)_{|m} = \hat{r}_{\Delta,m}^0. \tag{6.21}$$

Proof. The r-matrix $r_0^{m,\Delta}$ satisfies the equation

$$\tfrac{1}{2}\left(1 + \hat{h}\,\hat{s}\right)f = r_0^{m,\Delta}\left(1 - \hat{h}\,\hat{s}\right)f, \quad \forall f \in V_1^m. \tag{6.22}$$

Evidently, the operator $\hat{h}\,\hat{s}$ preserves $L\mathfrak{h}_m$, which is considered as a subspace of $L\mathfrak{h}^m$, hence (6.22) can be restricted to $L\mathfrak{h}_m$:

$$\tfrac{1}{2}\left(1 + \hat{h}\,\tau_m\right)f = \left(r_0^{m,\Delta}\right)_{|m}\left(1 - \hat{h}\,\tau_m\right)f, \quad \forall f \in V_m, \tag{6.23}$$

where $V_m = \left\{\mathrm{diag}\left(0, *, \ldots *\right) \subset L\mathfrak{h}_m\right\}$, $V_m = \left(V_1^m\right)_{|m}$. Obviously, $\left(r_0^{m,\Delta}\right)_{|m}$ is skew-symmetric; then, according to lemma 2.4, equation (6.23) implies that

$$\left(r_0^{m,\Delta}\right)_{|m} = \frac{1}{2}\frac{1+\hat{h}\tau_m}{1-\hat{h}\tau_m}\mathcal{P}_0' + m\tilde{\Delta}\mathcal{P}_{U_m} \equiv \hat{r}_{\tilde{\Delta},m}^0 \tag{6.24}$$

with some $\tilde{\Delta} \in \mathrm{End}U_m$.

To prove that $\Delta = \tilde{\Delta}$ let us calculate the bilinear form $\left\langle\left(r_0^{m,\Delta}\right)_{|m}f, g\right\rangle$, $f, g \in U_m$. For arbitrary $f \in U_m$ let us denote by $\hat{f}$ its image under the natural embedding $L\mathfrak{gl}_m \to L\mathfrak{gl}_q^m$.

Evidently,

$$\left\langle\left(r_0^{m,\Delta}\right)_{|m}f, g\right\rangle_{L\mathfrak{gl}_m} = \left\langle r_0^{m,\Delta}\hat{f}, \hat{g}\right\rangle_{L\mathfrak{gl}_q^m}. \tag{6.25}$$

By definition,

$$r_0^{m,\Delta} = -\tfrac{1}{2} + A^{-1} + \tfrac{m}{2}\mathcal{P}_U + m\left(\tfrac{1}{2}\frac{1+\hat{h}^m}{1-\hat{h}^m}\mathcal{P}_0' + \Delta\right)\mathcal{P}_U. \tag{6.26}$$

Using explicit formula (5.9) for A^{-1} we find:

$$\begin{aligned}
\left\langle\left(-\tfrac{1}{2} + A^{-1}\right)\hat{f}, \hat{g}\right\rangle &= -\int\frac{dz}{z}\sum_{l=0}^{m-1}\left(l + \tfrac{1}{2}\right)f_0\left(q^{-l}z\right)g_0\left(q^{-l}z\right) \\
&= -\frac{m^2}{2}\int\frac{dz}{z}f_0\left(z\right)g_0\left(z\right) = -\frac{m}{2}\left\langle f, g\right\rangle. \tag{6.27}
\end{aligned}$$

Then, from (5.9) it follows that

$$\left(\mathcal{P}_U \hat{f}\right)_{|m} = f; \tag{6.28}$$

indeed,

$$\left(\mathcal{P}_U \hat{f}\right)_0 (z) = \left[-\frac{1}{\lambda}\left(A^{-1}\hat{f}\right)\left(\lambda, q^\lambda z\right)\right]_{\lambda=m} = \left[-\frac{1}{\lambda}\left(-m\right) f_0 \left(q^{-\lambda} q^\lambda z\right)\right]_{\lambda=m} = f_0 \left(z\right).$$

The formulas (6.27) and (6.28) imply that

$$\left\langle \left(r_0^{m,\Delta}\right)_{|m} f, g \right\rangle = \left\langle r_0^{m,\Delta} \hat{f}, \hat{g} \right\rangle = \left\langle m\left(\frac{1}{2}\frac{1+\hat{h}^m}{1-\hat{h}^m}\mathcal{P}_0' + \Delta\right) f, g \right\rangle. \tag{6.29}$$

Recalling that by theorem 2.5

$$\left\langle \left(\frac{1}{2}\frac{1+\hat{h}\tau_m}{1-\hat{h}\tau_m}\mathcal{P}_0'\right) f, g \right\rangle = \left\langle \frac{m}{2}\frac{1+\hat{h}^m}{1-\hat{h}^m}\mathcal{P}_0' f, g \right\rangle, \quad \forall f, g \in U_m, \tag{6.30}$$

and comparing (6.24) with (6.29) we find $\tilde{\Delta} = \Delta$, as desired. ∎

References

[1] Adler M. *On a trace functional for formal pseudodifferential operators and the symplectic structure of the Korteweg-de Vries type equations.* Inv. Math. **50**(1979), 219–248.
[2] Drinfeld V.G., Sokolov V.V. *Lie algebras and equations of Korteweg-de Vries type.* Sov. Math. Dokl. **23** (1981), 457–62; J. Sov. Math. **30** (1985), 1975–2035.
[3] B.L.Feigin. *Lie algebras $\mathfrak{gl}(\lambda)$ and cohomology of a Lie algebra of differential operators.* Russian Mathematical Surveys **43**(1988), no. 2, p.169–170.
[4] Feigin B., Frenkel E. *Affine Lie algebras at the critical level and Gelfand-Dickey algebras.* Int. J. Math. Phys. **A7**, suppl.A1 (1992), 197–215.
[5] L.Freidel, J.M.Maillet , *Quadratic algebras and integrable systems*, Phys. Lett. **B 262,** (1991), p.278.
[6] E.Frenkel. *Deformations of the KDV hierarchy and related soliton equations.* Int. Math. Res. Notices **2**(1996) 55–76; q-alg/9511003.
[7] Frenkel E., Reshetikhin N. *Quantum affine algebras and deformations of the Virasoro algebra and $\mathcal{W}$-algebras.* Comm. Math. Phys. **178**(1996), 237–264; q-alg/9505025.
[8] E.Frenkel, N. Reshetikhin, M.A.Semenov-Tian-Shansky. *Drinfeld-Sokolov reduction for difference operators and deformations of W-algebras. I. The case of Virasoro algebra.* Commun. Math. Phys. **192** (1998), 605-629.
[9] Gelfand I.M., Dickey L.A. *A family of Hamiltonian structures related to nonlinear integrable differential equations.* Preprint no. 136, Inst. Appl. Math. USSR Acad. Sci. 1978 (in Russian), English transl. in: Collected papers of I. M. Gelfand, Vol. 1. Berlin, Heidelberg, New York : Springer 1987, pp. 625–646
[10] B.Khesin and I.Zakharevich. *Poisson-Lie group of pseudodifferential symbols and fractional KP-KdV hierarchies.* —C.R.Acad. Sci. Paris, **316**(1993), Serie I, p.621–626.
[11] B.Khesin and F.Malikov. *Universal Drinfeld-Sokolov Reduction and Matrices of Complex Size.* Preprint hep-th/9405116, Commun.Math.Phys. **175** (1996) 113.

[12] Li, L. C., Parmentier, S. *Nonlinear Poisson structures and r-matrices.* Commun. Math. Phys. **125** (1989), 545–563.

[13] W.Oewel. *Poisson brackets for integrable lattice systems.* Algebraic aspects of integrable systems: in memory of Irene Dorfman. Progress in non-linear differential equations and their applications, v.26. Ed. A.S.Fokas and I.M.Gelfand, 1997.

[14] A.L.Pirozerski and M.A.Semenov-Tian-Shansky.*Generalized q-deformed Gelfand-Dickey structures on the group of q-pseudodifference operators.* Preprint math.QA/9811025.

[15] M.A.Semenov-Tian-Shansky. *Dressing transformations and Poisson-Lie group actions,* — Publ. RIMS, Kyoto University **21, No.6**(1985), 1203 – 1221.

[16] M.A.Semenov-Tian-Shansky. *Monodromy Map and Classical r-matrices,* Zapiski Nauchn.Semin. POMI, v.200, 1993, St.Petersburg, preprint hep-th/9402054.

[17] M.A.Semenov-Tian-Shansky and A.V.Sevostyanov. *Drinfeld-Sokolov reduction for difference operators and deformations of W-algebras. II. General semisimple case.* Commun. Math. Phys. **192** (1998), 631.